

FREE HOLOMORPHIC FUNCTIONS ON THE UNIT BALL OF $B(\mathcal{H})^n$. II.

GELU POPESCU

ABSTRACT. In this paper we continue the study of free holomorphic functions on the noncommutative ball

$$[B(\mathcal{H})^n]_1 := \left\{ (X_1, \dots, X_n) \in B(\mathcal{H})^n : \|X_1 X_1^* + \dots + X_n X_n^*\|^{1/2} < 1 \right\},$$

where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space \mathcal{H} , and $n = 1, 2, \dots$ or $n = \infty$. Several classical results from complex analysis have free analogues in our noncommutative setting.

We prove a maximum principle, a Naimark type representation theorem, and a Vitali convergence theorem, for free holomorphic functions with operator-valued coefficients. We introduce the class of free holomorphic functions with the radial infimum property and study it in connection with factorizations and noncommutative generalizations of some classical inequalities obtained by Schwarz and Harnack. The Borel-Carathéodory theorem is extended to our noncommutative setting.

Using a noncommutative generalization of Schwarz's lemma and basic facts concerning the free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$, we obtain an analogue of Julia's lemma for free holomorphic functions $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1$. We also obtain Pick-Julia theorems for free holomorphic functions with operator-valued coefficients.

We provide a noncommutative generalization of a classical inequality due to Lindelöf, which turns out to be sharper than the noncommutative von Neumann inequality.

Finally, we introduce a pseudohyperbolic metric on $[B(\mathcal{H})^n]_1$ which is invariant under the action of the free holomorphic automorphism group of $[B(\mathcal{H})^n]_1$ and turns out to be a noncommutative extension of the pseudohyperbolic distance on \mathbb{B}_n , the open unit ball of \mathbb{C}^n . In this setting, we obtain a Schwarz-Pick type lemma.

We also provide commutative versions of these results for operator-valued multipliers of the Drury-Arveson space.

CONTENTS

Introduction

1. *Free holomorphic functions: fractional transforms, maximum principle, and geometric structure*
2. *Vitali convergence and identity theorem for free holomorphic functions*
3. *Free holomorphic functions with the radial infimum property*
4. *Factorizations and free holomorphic versions of classical inequalities*
5. *Noncommutative Borel-Carathéodory theorems*
6. *Julia's lemma for holomorphic functions on noncommutative balls*
7. *Pick-Julia theorems for free holomorphic functions with operator-valued coefficients*
8. *Lindelöf inequality and sharpened forms of the noncommutative von Neumann inequality*
9. *Pseudohyperbolic metric on the unit ball of $B(\mathcal{H})^n$ and an invariant Schwarz-Pick lemma*

References

Date: January 27, 2008.

2000 Mathematics Subject Classification. Primary: 46L52; 47A56; Secondary: 46G20; 47A63.

Key words and phrases. Multivariable operator theory; Noncommutative function theory; Free holomorphic functions; Fock space; Poisson transform; Schwarz's lemma; Julia's lemma; Pick's theorem; pseudohyperbolic metric, Borel-Carathéodory theorem; von Neumann inequality.

Research supported in part by an NSF grant.

INTRODUCTION

In this paper, we continue our program to develop a *noncommutative analytic function theory* on the unit ball of $B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space \mathcal{H} . Initiated in [39], the theory of free holomorphic (resp. pluriharmonic) functions on the unit ball of $B(\mathcal{H})^n$, with operator-valued coefficients, has been developed very recently (see [41], [42], [43], [44], [45], [46], and [47]) in the attempt to provide a framework for the study of arbitrary n -tuples of operators on a Hilbert space. Several classical results from complex analysis, hyperbolic geometry, and interpolation theory have free analogues in this noncommutative multivariable setting. Related to our work, we mention the papers [20], [22], [27], [28], and [55], where several aspects of the theory of noncommutative analytic functions are considered in various settings.

To put our work in perspective, we need to set up some notation and recall some definitions. Let \mathbb{F}_n^+ be the unital free semigroup on n generators g_1, \dots, g_n and the identity g_0 . The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, n\}$. If $(X_1, \dots, X_n) \in B(\mathcal{H})^n$, we set $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$, the identity on \mathcal{H} .

We defined the algebra $H_{\mathbf{ball}_\gamma}$ of free holomorphic functions on the open operatorial n -ball of radius $\gamma > 0$,

$$[B(\mathcal{H})^n]_\gamma := \left\{ (X_1, \dots, X_n) \in B(\mathcal{H})^n : \|X_1 X_n^* + \cdots + X_n X_1^*\|^{1/2} < \gamma \right\},$$

as the set of all power series $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ with radius of convergence $\geq \gamma$, i.e., $\{a_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ are complex numbers with $\limsup_{k \rightarrow \infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2 \right)^{1/2k} \leq \frac{1}{\gamma}$. A free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ is the representation of an element $F \in H_{\mathbf{ball}_\gamma}$ on the Hilbert space \mathcal{H} , that is, the mapping

$$[B(\mathcal{H})^n]_\gamma \ni (X_1, \dots, X_n) \mapsto F(X_1, \dots, X_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha \in B(\mathcal{H}),$$

where the convergence is in the operator norm topology. Due to the fact that a free holomorphic function is uniquely determined by its representation on an infinite dimensional Hilbert space, we identify, throughout this paper, a free holomorphic function with its representation on a separable infinite dimensional Hilbert space.

We recall that a free holomorphic function F on $[B(\mathcal{H})^n]_1$ is bounded if $\|F\|_\infty := \sup \|F(X)\| < \infty$, where the supremum is taken over all $X \in [B(\mathcal{H})^n]_1$ and \mathcal{H} is an infinite dimensional Hilbert space. Let $H_{\mathbf{ball}}^\infty$ be the set of all bounded free holomorphic functions and let $A_{\mathbf{ball}}$ be the set of all elements F such that the mapping

$$[B(\mathcal{H})^n]_1 \ni (X_1, \dots, X_n) \mapsto F(X_1, \dots, X_n) \in B(\mathcal{H})$$

has a continuous extension to the closed unit ball $[B(\mathcal{H})^n]_1^-$. We showed in [39] that $H_{\mathbf{ball}}^\infty$ and $A_{\mathbf{ball}}$ are Banach algebras under pointwise multiplication and the norm $\|\cdot\|_\infty$, which can be identified with the noncommutative analytic Toeplitz algebra F_n^∞ and the noncommutative disc algebra \mathcal{A}_n , respectively.

In Section 1, we present new results concerning the composition of free holomorphic functions with operator-valued coefficients and the behavior of their model boundary functions, which will play an important role throughout this paper.

Fractional maps of the operatorial unit ball $[B(\mathcal{E}, \mathcal{G})]_1^-$ are due to Siegel [52] and Phillips [30] (see also [57]). We should mention that the noncommutative ball $[B(\mathcal{H})^n]_1$ can be identified with the open unit ball of $B(\mathcal{H}^n, \mathcal{H})$, which is one of the infinite-dimensional Cartan domains studied by L. Harris ([19]). He has obtained several results, related to our topic, in the setting of JB^* -algebras. We also remark that the group of all free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$ ([45]), can be identified with a subgroup of the group of automorphisms of $[B(\mathcal{H}^n, \mathcal{H})]_1$ considered by R.S. Phillips [30] (see also [57]).

Following these ideas, fractional transforms of free holomorphic functions were recently considered in [43], [47], and [20]. In Section 1, we continue to investigate these transforms and work out several of their properties. A fractional transform Ψ_A is associated with each strict contraction $A = I \otimes A_0$, $A_0 \in B(\mathcal{E}, \mathcal{G})$. We show that $\Psi_A : \mathcal{S}_{\mathbf{ball}}(B(\mathcal{E}, \mathcal{G})) \rightarrow \mathcal{S}_{\mathbf{ball}}(B(\mathcal{E}, \mathcal{G}))$ defined by

$$\Psi_A[F] := A - D_{A^*}(I - FA^*)^{-1}FD_A$$

is a homeomorphism of the noncommutative Schur class $\mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G}))$ of all free holomorphic functions F on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E}, \mathcal{G})$ such that $\|F\|_\infty \leq 1$. Among other properties, we prove that F is inner if and only if its fractional transform $\Psi_A[F]$ is inner, and that the model boundary function \tilde{F} is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ if and only if $\widehat{\Psi_A[F]}$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$.

We mention that the noncommutative Schur class $\mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G}))$ was introduced in [34] in connection with a noncommutative von Neumann inequality for row contractions. This class was extended to more general settings by Ball-Groenewald-Malakorn (see [5], [6]), and by Muhly-Solel (see [26], [27], [28]). The Muhly-Solel paper [28] gives an intrinsic characterization for the Schur class $\mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G}))$ in terms of completely positive kernels, and presents a description of the automorphism group of their Hardy algebra $H^\infty(E)$, which has some overlap with [45] and Theorem 1.3 of the present paper.

Using fractional transforms and a noncommutative version of Schwarz's lemma [39], we prove a maximum principle for free holomorphic functions with operator-valued coefficients (see [45] for the scalar case). On the other hand, using fractional transforms, the noncommutative Cayley transforms of [41], and [44], we obtain results concerning the geometric structure of bounded free holomorphic functions. More precisely, we prove that a map $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ is a bounded free holomorphic function such that $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$, if and only if there exist a strict contraction $A_0 \in B(\mathcal{E})$, an n -tuple of isometries (V_1, \dots, V_n) on a Hilbert space \mathcal{K} , with orthogonal ranges, and an isometry $W : \mathcal{E} \rightarrow \mathcal{K}$, such that

$$F = (\Psi_{I \otimes A_0} \circ \mathcal{C})(G),$$

where \mathcal{C} is the noncommutative Cayley transform and G is defined by

$$G(X_1, \dots, X_n) := (I \otimes W^*) [2(I - X_1 \otimes V_1^* - \dots - X_n \otimes V_n^*)^{-1} - I] (I \otimes W)$$

for any $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$. In particular, in the scalar case, we obtain a characterization and parametrization of all bounded free holomorphic functions on the unit ball $[B(\mathcal{H})^n]_1$. We mention that, for the noncommutative polydisc, a representation theorem of the same flavor was obtained in [22] and [1].

In Section 2, we provide a Vitali type convergence theorem [21] for uniformly bounded sequences of free holomorphic functions with operator-valued coefficients. As a consequence, we show that two free holomorphic functions F, G coincide if and only if there exists a sequence $\{A^{(k)}\}_{k=1}^\infty \subset [B(\mathcal{H})^n]_1$ of bounded-below operators such that $\lim_{k \rightarrow \infty} \|A^{(k)}\| = 0$ and $F(A^{(k)}) = G(A^{(k)})$ for any $k = 1, 2, \dots$

In Section 3, we introduce the class of free holomorphic functions with the *radial infimum property*. A function F is in this class if

$$\liminf_{r \rightarrow 1} \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\| = \|F\|_\infty,$$

where S_1, \dots, S_n are the left creation operators on the full Fock space $F^2(H_n)$ with n generators. We obtain several characterizations for this class of functions and consider several examples. We show that if F is inner and its boundary function \tilde{F} is in the noncommutative disc algebra \mathcal{A}_n then F has the radial infimum property. In particular, any free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ has the property. We study the radial infimum property in connection with products, direct sums, and compositions of free holomorphic functions. We also show that the class of functions with the radial infimum property is invariant under the fractional transforms of Section 1. These results are important in the following sections.

It is well-known that if $f \in H^\infty(\mathbb{D})$, a bounded analytic function on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, is such that $\|f\|_\infty \leq 1$ and

$$f(z) = \theta(z)g(z), \quad z \in \mathbb{D},$$

where θ is an inner function in the disc algebra $A(\mathbb{D})$ and g is analytic in \mathbb{D} , then $\|g\|_\infty \leq 1$. If, in addition, $f \in A(\mathbb{D})$, then $g \in A(\mathbb{D})$. Moreover, if $f \in A(\mathbb{D})$ is inner, then so is g . These facts are fundamental for the theory of bounded analytic functions (see [9], [16]).

In Section 4, we obtain analogues of these results in the context of free holomorphic functions. Let F, Θ , and G be free holomorphic functions on $[B(\mathcal{H})^n]_1$ such that

$$F(X) = \Theta(X)G(X), \quad X \in [B(\mathcal{H})^n]_1.$$

Assume that F is bounded with $\|F\|_\infty \leq 1$ and Θ has the radial infimum property with $\|\Theta\|_\infty = 1$. Then we prove that $\|G\|_\infty \leq 1$ and

$$F(X)F(X)^* \leq \Theta(X)\Theta(X)^*, \quad X \in [B(\mathcal{H})^n]_1.$$

Moreover, we show that if the boundary functions \tilde{F} and $\tilde{\Theta}$ are in the noncommutative disc algebra, then so is \tilde{G} . When we add the condition that F is inner, then we deduce that G is also inner. In particular, if F is a bounded free holomorphic function with $\|F\|_\infty \leq 1$ and representation

$$F(X) = \sum_{k=m}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)}, \quad X = (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

for some $m = 1, 2, \dots$, and $A_{(\alpha)} \in B(\mathcal{E}, \mathcal{G})$, then

$$F(X)F(X)^* \leq \sum_{|\beta|=m} X_\beta X_\beta^* \otimes I_{\mathcal{G}}, \quad X \in [B(\mathcal{H})^n]_1.$$

Consequently, we recover the corresponding version of Schwarz's lemma from [39] and, when $m = 1$, the one from [20].

The classical Schwarz's lemma (see [10], [50]) states that if $f : \mathbb{D} \rightarrow \mathbb{C}$ is a bounded analytic function with $f(0) = 0$ and $|f(z)| \leq 1$ for $z \in \mathbb{D}$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for $z \in \mathbb{D}$. Moreover, if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \neq 0$, then there is a constant c with $|c| = 1$ such that $f(w) = cw$ for any $w \in \mathbb{D}$. A faithful generalization of this result is obtained (see Theorem 4.5) when f, θ , and g are free holomorphic functions on $[B(\mathcal{H})^n]_1$ with scalar coefficients such that:

- (i) $f(X) = \theta(X)g(X)$, $X \in [B(\mathcal{H})^n]_1$;
- (ii) f is bounded with $\|f\|_\infty \leq 1$;
- (iii) θ has the radial infimum property and $\|\theta\|_\infty = 1$.

In the particular case when $n = 1$ and $\theta(z) = z$, we recover the Schwarz's lemma. We remark that Schwarz's lemma has been extended to various settings by several authors (e.g. [25], [52], [49], [30], [18], [14], [27], [46], [45]).

In Section 4, we also obtain noncommutative extensions of Harnack' double inequality (see Theorem 4.9) for a class of free holomorphic functions $F = I + \Theta\Gamma$ with positive real parts. In the particular case when $\Theta(X) = X$, we deduce that if F is a free holomorphic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ such that $F(0) = I$ and $\Re F \geq 0$, then

$$\frac{1 - \|X\|}{1 + \|X\|} \leq \|F(X)\| \leq \frac{1 + \|X\|}{1 - \|X\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

The Borel-Carathéodory theorem [53] establishes an upper bound for the modulus of a function on the circle $|z| = r$ from bounds for its real (or imaginary) parts on larger circles $|z| = R$. More precisely, if f is an analytic function for $|z| \leq R$ and $0 < r < R$, then

$$\sup_{|z|=r} |f(z)| \leq \frac{2r}{R-r} \sup_{|z|=R} \Re f(z) + \frac{R+r}{R-r} |f(0)|.$$

In Section 5, we obtain an analogue of this result for free holomorphic functions (see Theorem 5.4). We also obtain a Borel-Carathéodory type result for free holomorphic functions which admit factorizations $F = \Theta\Gamma$, where Θ is an inner function with the radial infimum property and $\|\Theta(0)\| < 1$. We show that if $\Re F \leq I$ then

$$\|F(X)\| \leq \frac{2\|\Theta(X)\|}{1 - \|\Theta(X)\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

Let $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ be a nonconstant analytic function and let $z_0 \in \mathbb{D}$ and $w_0 = f(z_0)$. Pick's theorem [31] (see also [9]) asserts that

$$\frac{w_0 - f(z)}{1 - \bar{w}_0 f(z)} = \frac{z_0 - z}{1 - \bar{z}_0 z} g(z), \quad z \in \mathbb{D},$$

for some analytic function $g : \mathbb{D} \rightarrow \mathbb{D}$. In Section 6, we provide a generalization of Pick's theorem, for bounded free holomorphic functions. We show that if $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ is a free holomorphic function with $\|F(0)\| < 1$ and $a \in \mathbb{B}_n$, then there exists a free holomorphic function Γ with $\|\Gamma\|_\infty \leq 1$ such that

$$\Phi_{F(a)}(F(X)) = \Phi_a(X)(\Gamma \circ \Phi_a)(X), \quad X \in [B(\mathcal{H})^n]_1,$$

where Φ_a and $\Phi_{F(a)}$ are the corresponding free holomorphic automorphisms of the noncommutative balls $[B(\mathcal{H})^n]_1$ and $[B(\mathcal{H})^m]_1$, respectively. Consequently,

$$\Phi_{F(a)}(F(X))\Phi_{F(a)}(F(X))^* \leq \Phi_a(X)\Phi_a(X)^*, \quad X \in [B(\mathcal{H})^n]_1.$$

We mention that the group $\text{Aut}([B(\mathcal{H})^n]_1)$ of all free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$ was determined in [45], using the theory of characteristic functions for row contractions [33]. We also remark that the group of all free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$, can be identified with a subgroup of the group of automorphisms of $[B(\mathcal{H}^n, \mathcal{H})]_1$ considered by R.S. Phillips [30] (see also [57]).

We recall that Julia's lemma [25] (see also [8]) says that if $f : \mathbb{D} \rightarrow \mathbb{D}$ is an analytic function and there is a sequence $\{z_k\} \subset \mathbb{D}$ with $z_k \rightarrow 1$, $f(z_k) \rightarrow 1$, and such that $\frac{1-|f(z_k)|}{1-|z_k|}$ is bounded, then f maps each disc in \mathbb{D} tangent to $\partial\mathbb{D}$ at 1 into a disc of the same kind. Julia's lemma has been extended to analytic functions of a single operator variable by Fan [15] and to the setting of function algebras by Glicksberg [17].

Using the above-mentioned noncommutative analogue of Pick's theorem and basic facts concerning the involutive free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$, we obtain a free analogue of Julia's lemma (see Theorem 6.3). In particular, we prove the following result.

Let $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1$ be a free holomorphic function. Let $z_k \in \mathbb{B}_n$ be such that $\lim_{k \rightarrow \infty} z_k = (1, 0, \dots, 0) \in \partial\mathbb{B}_n$, $\lim_{k \rightarrow \infty} F(z_k) = (1, 0, \dots, 0) \in \partial\mathbb{B}_m$, and

$$\lim_{k \rightarrow \infty} \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} = L < \infty.$$

If $F := (F_1, \dots, F_m)$, then $L > 0$ and

$$(I - F_1(X)^*)(I - F(X)F(X)^*)^{-1}(I - F_1(X)) \leq L(I - X_1^*)(I - XX^*)^{-1}(I - X_1)$$

for any $X = (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$. Moreover, if $0 < c < 1$, then

$$F(\mathbf{E}_c) \subset \mathbf{E}_\gamma, \quad \text{where } \gamma := \frac{Lc}{1 + Lc - c}$$

and \mathbf{E}_c and \mathbf{E}_γ are certain noncommutative ellipsoids. A similar result holds if we replace the ellipsoids with some noncommutative Korany type regions ([51]) in the unit ball $[B(\mathcal{H})^n]_1$ (see Corollary 6.5).

In Section 7, we use fractional transforms and a version of the noncommutative Schwarz's lemma to obtain Pick-Julia theorems for free holomorphic functions F with operator-valued coefficients such that $\|F\|_\infty \leq 1$ (resp. $\Re F \geq 0$) (see Theorem 7.1 and Theorem 7.2). As a consequence, we obtain a Julia type lemma for free holomorphic functions with positive real parts (see Theorem 6.4). We also provide commutative versions of these results for operator-valued multipliers of the Drury-Arveson space (see Corollary 7.4). When $n = 1$, we recover (with different proofs) the corresponding results obtained by Potapov [49] and Ando-Fan [2].

In Section 8, we provide a noncommutative extension of a classical result due to Lindelöf (see [16], [24]). We prove that if $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ is a free holomorphic function, then

$$\|F(X)\| \leq \frac{\|X\| + \|F(0)\|}{1 + \|X\|\|F(0)\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

If, in addition, the boundary function of F has its entries in the noncommutative disc algebra \mathcal{A}_n , then the inequality above holds for any $X \in [B(\mathcal{H})^n]_1^-$. We remark that if $\|F(0)\| < 1$, then the inequality above is sharper than the noncommutative von Neumann inequality (see [34], [35]).

In Section 9, we introduce a pseudohyperbolic metric \mathbf{d} on $[B(\mathcal{H})^n]_1$ which is invariant under the action of the free holomorphic automorphism group of $[B(\mathcal{H})^n]_1$ and turns out to be a noncommutative extension of the pseudohyperbolic distance (see [58]) on \mathbb{B}_n , the open unit ball of \mathbb{C}^n , i.e.,

$$d_n(z, w) := \|\psi_z(w)\|_2, \quad z, w \in \mathbb{B}_n,$$

where ψ_z is the involutive automorphism of \mathbb{B}_n that interchanges 0 and z . We show that

$$\mathbf{d}(X, Y) = \tanh \delta(X, Y), \quad X, Y \in [B(\mathcal{H})^n]_1,$$

where δ is the hyperbolic (*Poincaré-Bergman* [7] type) metric on $[B(\mathcal{H})^n]_1$ introduced and studied in [47]. As a consequence, we obtain a Schwarz-Pick lemma for free holomorphic functions on $[B(\mathcal{H})^n]_1$ with operator-valued coefficients, with respect to the pseudohyperbolic metric. More precisely, if $F = (F_1, \dots, F_m)$ and F_j are free holomorphic functions with operator-valued coefficients such that $\|F\|_\infty \leq 1$, then

$$\mathbf{d}(F(X), F(Y)) \leq \mathbf{d}(X, Y), \quad X, Y \in [B(\mathcal{H})^n]_1.$$

It is well-known (see [37], [39], [44]) that if F is a contractive ($\|F\|_\infty \leq 1$) free holomorphic function with coefficients in $B(\mathcal{E})$, then the evaluation map $\mathbb{B}_n \ni z \mapsto F(z) \in B(\mathcal{E})$ is a contractive operator-valued multiplier of the Drury-Arveson space ([13], [3]). Moreover, any such a contractive multiplier has this kind of representation. Due to this reason, several results of the present paper have commutative versions for operator-valued multipliers of the Drury-Arveson space.

It would be interesting to see if the results of this paper can be extended to more general infinite-dimensional bounded domains such as the JB^* -algebras of Harris [19], or the noncommutative domains from [48] and [20]. Since our results are based on the power series representation of free holomorphic functions we are inclined to believe in a positive answer for the domains considered in [48] and [20].

1. FREE HOLOMORPHIC FUNCTIONS: FRACTIONAL TRANSFORMS, MAXIMUM PRINCIPLE, AND GEOMETRIC STRUCTURE

In this section, we present results concerning the composition and fractional transforms of free holomorphic functions, and the behavior of their model boundary functions. These results are used to prove a maximum principle and a Naimark type representation theorem for free holomorphic functions with operator-valued coefficients.

Let H_n be an n -dimensional complex Hilbert space with orthonormal basis e_1, e_2, \dots, e_n , where $n = 1, 2, \dots$, or $n = \infty$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k \geq 1} H_n^{\otimes k},$$

where $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n . Define the left (resp. right) creation operators S_i (resp. R_i), $i = 1, \dots, n$, acting on $F^2(H_n)$ by setting

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n),$$

(resp. $R_i \varphi := \varphi \otimes e_i$, $\varphi \in F^2(H_n)$). The noncommutative disc algebra \mathcal{A}_n (resp. \mathcal{R}_n) is the norm closed algebra generated by the left (resp. right) creation operators and the identity. The noncommutative analytic Toeplitz algebra F_n^∞ (resp. \mathcal{R}_n^∞) is the weakly closed version of \mathcal{A}_n (resp. \mathcal{R}_n). These algebras were introduced in [34] in connection with a noncommutative von Neumann type inequality [56], and have been intensively studied in recent years (see [35], [36], [37], [38], [12], [46], [26], and the references therein).

We denote $e_\alpha := e_{i_1} \otimes \dots \otimes e_{i_k}$ if $\alpha = g_{i_1} \dots g_{i_k} \in \mathbb{F}_n^+$ and $e_{g_0} := 1$. Note that $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is an orthonormal basis for $F^2(H_n)$. Let $C^*(S_1, \dots, S_n)$ be the Cuntz-Toeplitz C^* -algebra generated by the left

creation operators (see [11]). The noncommutative Poisson transform at $T := (T_1, \dots, T_n) \in [B(\mathcal{H})^n]_1^-$ is the unital completely contractive linear map $P_T : C^*(S_1, \dots, S_n) \rightarrow B(\mathcal{H})$ defined by

$$P_T[f] := \lim_{r \rightarrow 1} K_{T,r}^*(I_{\mathcal{H}} \otimes f)K_{T,r}, \quad f \in C^*(S_1, \dots, S_n),$$

where the limit exists in the norm topology of $B(\mathcal{H})$. Here, the noncommutative Poisson kernel

$$K_{T,r} : \mathcal{H} \rightarrow \overline{\Delta_{T,r}\mathcal{H}} \otimes F^2(H_n), \quad 0 < r \leq 1,$$

is defined by

$$K_{T,r}h := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} \Delta_{T,r} T_{\alpha}^* h \otimes e_{\alpha}, \quad h \in \mathcal{H},$$

where $\Delta_{T,r} := (I_{\mathcal{H}} - r^2 T_1 T_1^* - \dots - r^2 T_n T_n^*)^{1/2}$ and $\Delta_T := \Delta_{T,1}$. We recall that

$$P_T[S_{\alpha} S_{\beta}^*] = T_{\alpha} T_{\beta}^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

When $T := (T_1, \dots, T_n)$ is a pure row contraction, i.e., $\text{SOT-}\lim_{k \rightarrow \infty} \sum_{|\alpha|=k} T_{\alpha} T_{\alpha}^* = 0$, then we have

$$P_T[f] = K_T^*(I_{\mathcal{D}_T} \otimes f)K_T, \quad f \in C^*(S_1, \dots, S_n) \text{ or } f \in F_n^{\infty},$$

where $\mathcal{D}_T := \overline{\Delta_T \mathcal{H}}$. We refer to [37], [38], and [46] for more on noncommutative Poisson transforms on C^* -algebras generated by isometries.

Let \mathcal{E}, \mathcal{G} be Hilbert spaces and let $B(\mathcal{E}, \mathcal{G})$ be the set of all bounded linear operators from \mathcal{E} to \mathcal{G} . A map $F : [B(\mathcal{H})^n]_{\gamma} \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is a *free holomorphic function* on $[B(\mathcal{H})^n]_{\gamma}$ with coefficients in $B(\mathcal{E}, \mathcal{G})$ if there exist $A_{(\alpha)} \in B(\mathcal{E}, \mathcal{G})$, $\alpha \in \mathbb{F}_n^+$, such that

$$F(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_{\alpha} \otimes A_{(\alpha)},$$

where the series converges in the operator norm topology for any $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_{\gamma}$. According to [39], a power series $F := \sum_{\alpha \in \mathbb{F}_n^+} Z_{\alpha} \otimes A_{(\alpha)}$ represents a free holomorphic function on the open operatorial

n -ball of radius γ , with coefficients in $B(\mathcal{E}, \mathcal{G})$, if and only if $\limsup_{k \rightarrow \infty} \left\| \sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \right\|^{\frac{1}{2k}} \leq \frac{1}{\gamma}$. This is also equivalent to the fact that the series

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} S_{\alpha} \otimes A_{(\alpha)}$$

is convergent in the operator norm topology for any $r \in [0, \gamma)$, where S_1, \dots, S_n are the left creation operators on the Fock space $F^2(H_n)$. We denote by $H_{\mathbf{ball}}(B(\mathcal{E}, \mathcal{G}))$ the set of all free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_1$ and coefficients in $B(\mathcal{E}, \mathcal{G})$. Let $H_{\mathbf{ball}}^{\infty}(B(\mathcal{E}, \mathcal{G}))$ denote the set of all elements F in $H_{\mathbf{ball}}(B(\mathcal{E}, \mathcal{G}))$ such that

$$\|F\|_{\infty} := \sup \|F(X_1, \dots, X_n)\| < \infty,$$

where the supremum is taken over all n -tuples of operators $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$, where \mathcal{H} is an infinite dimensional Hilbert space.

According to [39] and [44], the noncommutative Hardy space $H_{\mathbf{ball}}^{\infty}(B(\mathcal{E}, \mathcal{G}))$ can be identified to the operator space $F_n^{\infty} \bar{\otimes} B(\mathcal{E}, \mathcal{G})$ (the weakly closed operator space generated by the spatial tensor product), where F_n^{∞} is the noncommutative analytic Toeplitz algebra. More precisely, a bounded free holomorphic function F is uniquely determined by its (model) boundary function $\tilde{F}(S_1, \dots, S_n) \in F_n^{\infty} \bar{\otimes} B(\mathcal{E}, \mathcal{G})$ defined by

$$\tilde{F} = \tilde{F}(S_1, \dots, S_n) := \text{SOT-}\lim_{r \rightarrow 1} F(rS_1, \dots, rS_n).$$

Moreover, F is the noncommutative Poisson transform of $\tilde{F}(S_1, \dots, S_n)$ at $X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$, i.e.,

$$F(X_1, \dots, X_n) = (P_X \otimes I)[\tilde{F}(S_1, \dots, S_n)].$$

Similar results hold for bounded free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_{\gamma}$, $\gamma > 0$.

We recall from [45] some facts concerning the composition of free holomorphic functions with operator-valued coefficients. Let $\Phi : [B(\mathcal{H})^n]_{\gamma_1} \rightarrow [B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{Y})]^m$ be a free holomorphic function with $\Phi(X) := (\Phi_1(X), \dots, \Phi_m(X))$, where $\Phi_j : [B(\mathcal{H})^n]_{\gamma_1} \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{Y})$, $j = 1, \dots, m$, are free holomorphic functions with standard representations

$$\Phi_j(X) = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha|=k} X_{\alpha} \otimes B_{(\alpha)}^{(j)}, \quad X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_{\gamma_2},$$

for some $B_{(\alpha)}^{(j)} \in B(\mathcal{Y})$, where $\alpha \in \mathbb{F}_n^+$, $j = 1, \dots, m$. Assume that

$$\|\Phi(X)\| < \gamma_2 \quad \text{for any } X \in [B(\mathcal{H})^n]_{\gamma_1}.$$

This is equivalent to

$$\|\Phi(rS_1, \dots, rS_n)\| < \gamma_2 \quad \text{for any } r \in [0, \gamma_1).$$

Let $F : [B(\mathcal{K})^m]_{\gamma_2} \rightarrow B(\mathcal{K}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a free holomorphic function with standard representation

$$F(Y_1, \dots, Y_m) := \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} Y_{\alpha} \otimes A_{(\alpha)}, \quad (Y_1, \dots, Y_m) \in [B(\mathcal{K})^m]_{\gamma_2},$$

for some operators $A_{(\alpha)} \in B(\mathcal{E}, \mathcal{G})$, $\alpha \in \mathbb{F}_m^+$. Then it makes sense to define the map $F \circ \Phi : [B(\mathcal{H})^n]_{\gamma_1} \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{Y}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ by setting

$$(F \circ \Phi)(X_1, \dots, X_n) := \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} \Phi_{\alpha}(X_1, \dots, X_n) \otimes A_{(\alpha)}, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_{\gamma_1},$$

where the convergence is in the operator norm topology. We proved in [45] that $F \circ \Phi$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$ with standard representation

$$(F \circ \Phi)(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{\sigma \in \mathbb{F}_n^+, |\sigma|=k} X_{\sigma} \otimes C_{(\sigma)},$$

where

$$\langle C_{(\sigma)}x, y \rangle = \frac{1}{r^{|\sigma|}} \langle (S_{\sigma}^* \otimes I_{\mathcal{Y} \otimes \mathcal{G}})(F \circ \Phi)(rS_1, \dots, rS_n)(1 \otimes x), 1 \otimes y \rangle$$

for any $\sigma \in \mathbb{F}_n^+$, $x \in \mathcal{Y} \otimes \mathcal{E}$, and $y \in \mathcal{Y} \otimes \mathcal{G}$. Actually, this is a slight extension of the corresponding result from [45]. However, the proof is basically the same.

For simplicity, throughout this paper, $[X_1, \dots, X_n]$ denotes either the n -tuple $(X_1, \dots, X_n) \in B(\mathcal{H})^n$ or the operator row matrix $[X_1 \cdots X_n]$ acting from $\mathcal{H}^{(n)}$, the direct sum of n copies of a Hilbert space \mathcal{H} , to \mathcal{H} .

Now, we present new results concerning the composition of bounded free holomorphic functions with operator-valued coefficients.

Theorem 1.1. *Let $F : [B(\mathcal{K})^m]_{\gamma_2} \rightarrow B(\mathcal{K}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ and $\Phi : [B(\mathcal{H})^n]_{\gamma_1} \rightarrow [B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{Y})]^m$ be bounded free holomorphic functions such that*

$$\|\Phi(X)\| < \gamma_2 \quad \text{for any } X \in [B(\mathcal{H})^n]_{\gamma_1}.$$

Then the boundary function of the bounded free holomorphic function $F \circ \Phi$ satisfies the equation

$$\widetilde{F \circ \Phi} = \text{SOT-} \lim_{r \rightarrow 1} F(r\tilde{\Phi}_1, \dots, r\tilde{\Phi}_m).$$

Moreover, if $\tilde{F} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ and $\tilde{\Phi} := [\tilde{\Phi}_1, \dots, \tilde{\Phi}_m]$ is such that $\tilde{\Phi}_j \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y})$, $j = 1, \dots, m$, and $\|\tilde{\Phi}\| < \gamma_2$, then $\widetilde{F \circ \Phi} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y} \otimes \mathcal{E}, \mathcal{Y} \otimes \mathcal{G})$.

Proof. Using the fact that a function $X \mapsto G(X)$ is free holomorphic on $[B(\mathcal{K})^m]_\gamma$, $\gamma > 0$, if and only if the mapping $Y \mapsto G(\gamma Y)$ is free holomorphic on $[B(\mathcal{K})^m]_1$, we can assume, without loss of generality, that $\gamma_1 = \gamma_2 = 1$.

Due to [45] (see the considerations preceding this theorem), $F \circ \Phi$ is a bounded free holomorphic function. Let F have the representation

$$F(Y_1, \dots, Y_m) := \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} Y_\alpha \otimes A_{(\alpha)}, \quad (Y_1, \dots, Y_m) \in [B(\mathcal{K})^m]_1.$$

Since F is bounded on $[B(\mathcal{K})^n]_1$, we have

$$\left(\sum_{\beta \in \mathbb{F}_m^+} \|A_{(\beta)} h\|^2 \right)^{1/2} \leq \|F\|_\infty \|h\|, \quad h \in \mathcal{E}.$$

Given $\epsilon > 0$ and $h \in \mathcal{E}$, we choose $q \in \mathbb{N}$ such that

$$(1.1) \quad \sum_{\beta \in \mathbb{F}_m^+, |\beta| \geq q} \|A_{(\beta)} h\|^2 < \epsilon^2.$$

For any $x \in F^2(H_n)$, we have

$$\begin{aligned} & \left\| \sum_{k=q}^{\infty} \sum_{\beta \in \mathbb{F}_m^+, |\beta|=k} (\Phi_\beta(rS_1, \dots, rS_n) \otimes A_{(\beta)})(x \otimes h) \right\| \\ & \leq \sum_{k=q}^{\infty} \left\| [\Phi_\beta(rS_1, \dots, rS_n) \otimes I : |\beta|=k] \begin{bmatrix} I \otimes A_{(\beta)} \\ \vdots \\ I \otimes A_{(\beta)} \end{bmatrix} (x \otimes h) \right\| \\ & \leq \sum_{k=q}^{\infty} \left\| \begin{bmatrix} I \otimes A_{(\beta)} \\ \vdots \\ I \otimes A_{(\beta)} \end{bmatrix} (x \otimes h) \right\| \\ & \leq \|x\| \sum_{k=q}^{\infty} \left(\sum_{\beta \in \mathbb{F}_m^+, |\beta|=k} \|A_{(\beta)} h\|^2 \right)^{1/2} \leq \epsilon \|x\| \end{aligned}$$

for any $r \in (0, 1)$. Here we used the fact that $[\Phi_1(rS_1, \dots, rS_n), \dots, \Phi_n(rS_1, \dots, rS_n)]$ is a contraction and, therefore, the operator row matrix $[\Phi_\beta(rS_1, \dots, rS_n) \otimes I : |\beta|=k]$ is also a contraction.

Now denote $F_r(Y_1, \dots, Y_m) := F(rY_1, \dots, rY_m)$, $0 < r < 1$, and note that F_r is a bounded free holomorphic function on $[B(\mathcal{K})^n]_{1/r}$. Since the boundary function $\tilde{\Phi} := [\tilde{\Phi}_1, \dots, \tilde{\Phi}_m]$ is a row contraction, we have

$$F_r(\tilde{\Phi}_1, \dots, \tilde{\Phi}_m) = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} r^{|\alpha|} \tilde{\Phi}_\alpha \otimes A_{(\alpha)},$$

where the convergence is in the operator norm topology.

Using relation (1.1) and that $[r^{|\beta|} \tilde{\Phi}_\beta \otimes I : |\beta|=k]$ is a row contraction, one can show, as above, that

$$\left\| \sum_{k=q}^{\infty} \sum_{\beta \in \mathbb{F}_m^+, |\beta|=k} (r^{|\beta|} \tilde{\Phi}_\beta \otimes A_{(\beta)})(x \otimes h) \right\| < \epsilon \|x\|$$

for any $r \in (0, 1)$. On the other hand, we have

$$\lim_{r \rightarrow 1} \sum_{\beta \in \mathbb{F}_m^+, |\beta| < k} [(\Phi_\beta(rS_1, \dots, rS_n) - r^{|\beta|} \tilde{\Phi}_\beta) \otimes A_{(\beta)}](x \otimes h) = 0.$$

Now, combining this equality with the inequalities above, one can easily deduce that

$$(1.2) \quad \lim_{r \rightarrow 1} (F \circ \Phi)(rS_1, \dots, rS_n)(x \otimes h) = \lim_{r \rightarrow 1} F(r\tilde{\Phi}_1, \dots, r\tilde{\Phi}_m)(x \otimes h)$$

for any $x \in F^2(H_n)$ and $h \in \mathcal{E}$. Since

$$\|F(r\tilde{\Phi}_1, \dots, r\tilde{\Phi}_m)\| \leq \|F\|_\infty \quad \text{and} \quad \|(F \circ \Phi)(rS_1, \dots, rS_n)\| \leq \|F\|_\infty,$$

relation (1.2) implies

$$\widetilde{F \circ \Phi} = \text{SOT-}\lim_{r \rightarrow 1} F(r\tilde{\Phi}_1, \dots, r\tilde{\Phi}_m).$$

To prove the second part of the theorem, assume that $\tilde{F} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$, $\tilde{\Phi} := [\tilde{\Phi}_1, \dots, \tilde{\Phi}_m]$ is in $M_{1 \times m}(\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y}))$, and $\|\tilde{\Phi}\| < 1$. Since F is a free holomorphic function on $[B(\mathcal{K})^n]_1$,

$$G := \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} \tilde{\Phi}_\alpha \otimes A_{(\alpha)}$$

is convergent in the operator norm topology. On the other hand, $\tilde{\Phi}_\alpha \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y})$. Consequently, G is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y} \otimes \mathcal{E}, \mathcal{Y} \otimes \mathcal{G})$. Now, for any $\epsilon > 0$, there exists $p \in \mathbb{N}$ such that

$$\left\| \sum_{k=p}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} \tilde{\Phi}_\alpha \otimes A_{(\alpha)} \right\| < \epsilon.$$

Due to the noncommutative von Neumann inequality (see [34]), we have

$$\left\| \sum_{k=p}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} \Phi_\alpha(rS_1, \dots, rS_n) \otimes A_{(\alpha)} \right\| \leq \left\| \sum_{k=p}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} \tilde{\Phi}_\alpha \otimes A_{(\alpha)} \right\|$$

for any $k \in \mathbb{N}$. Consequently, we have

$$(1.3) \quad \|(F \circ \Phi)(rS_1, \dots, rS_n) - G\| \leq \sum_{k=0}^p \left\| \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} [\Phi_\alpha(rS_1, \dots, rS_n) - \tilde{\Phi}_\alpha] \otimes A_{(\alpha)} \right\| + 2\epsilon$$

On the other hand, since $\tilde{\Phi}_i \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y})$, $i = 1, \dots, n$, we have

$$\lim_{r \rightarrow 1} \Phi_\alpha(rS_1, \dots, rS_n) = \tilde{\Phi}_\alpha, \quad \alpha \in \mathbb{F}_m^+,$$

in the operator norm topology. Now, using relation (1.3), we deduce that

$$\widetilde{F \circ \Phi} := \lim_{r \rightarrow 1} (F \circ \Phi)(rS_1, \dots, rS_n) = G,$$

where the limit is in the operator norm topology. Therefore $\widetilde{F \circ \Phi}$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y} \otimes \mathcal{E}, \mathcal{Y} \otimes \mathcal{G})$. This completes the proof. \square

Using Theorem 1.1, and Theorem 4.1 from [29], we can prove the following result for bounded free holomorphic functions with operator-valued coefficients. We recall that a bounded free holomorphic function is called inner (resp. outer) if its model boundary function is an isometry (resp. has dense range).

Theorem 1.2. *Let $F : [B(\mathcal{K})^m]_1 \rightarrow B(\mathcal{K}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ and $\Phi : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})]^m$ be bounded free holomorphic functions. Assume that $\Phi = (\Phi_1, \dots, \Phi_m)$ is inner and $\tilde{\Phi}_1$ is non-unitary if $m = 1$. Then the following statements hold:*

- (a) $\|F \circ \Phi\|_\infty = \|F\|_\infty$;
- (b) if F is inner, then $F \circ \Phi$ is inner;
- (c) if F is outer, then $F \circ \Phi$ is outer.

Proof. Let $\Phi_j : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$, $j = 1, \dots, m$, be free holomorphic functions with scalar coefficients and assume that $\Phi = [\Phi_1, \dots, \Phi_n]$ is inner, i.e., $\tilde{\Phi} := [\tilde{\Phi}_1, \dots, \tilde{\Phi}_n]$ is an isometry. According to Theorem 4.1 from [29], $\tilde{\Phi}$ is a pure isometry, i.e.,

$$\text{WOT-} \lim_{k \rightarrow \infty} \sum_{\omega \in \mathbb{F}_m^+, |\omega|=k} \tilde{\Phi}_\omega \tilde{\Phi}_\omega^* = 0.$$

Due to the noncommutative Wold-type decomposition for sequences of isometries with orthogonal ranges [32], $\tilde{\Phi}$ is unitarily equivalent to $[I_{\mathcal{L}} \otimes S'_1, \dots, I_{\mathcal{L}} \otimes S'_m]$, where S'_1, \dots, S'_m are the left creation operators on the full Fock space $F^2(H_m)$, and \mathcal{L} is a separable Hilbert space. Consequently, there is a unitary operator $U : F^2(H_n) \rightarrow \mathcal{L} \otimes F^2(H_m)$ such that

$$U \tilde{\Phi}_j = (I_{\mathcal{L}} \otimes S'_j)U, \quad j = 1, \dots, m.$$

Hence, if F has the representation

$$F(Y_1, \dots, Y_m) := \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} Y_\alpha \otimes A_{(\alpha)}, \quad (Y_1, \dots, Y_m) \in [B(\mathcal{K})^m]_1,$$

we deduce that

$$\begin{aligned} (U \otimes I)F(r\tilde{\Phi}_1, \dots, r\tilde{\Phi}_m) &= \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} r^{|\alpha|} (U \otimes I) \tilde{\Phi}_\alpha \otimes A_{(\alpha)} \\ &= \left(\sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m^+, |\alpha|=k} r^{|\alpha|} (I_{\mathcal{L}} \otimes S'_\alpha) \otimes A_{(\alpha)} \right) (U \otimes I) \\ &= [I_{\mathcal{L}} \otimes F(rS'_1, \dots, rS'_m)](U \otimes I). \end{aligned}$$

Now, using Theorem 1.1, we obtain

$$\begin{aligned} (1.4) \quad (U \otimes I)(\widetilde{F \circ \Phi}) &= (U \otimes I) \left(\text{SOT-} \lim_{r \rightarrow 1} F(r\tilde{\Phi}_1, \dots, r\tilde{\Phi}_m) \right) \\ &= \text{SOT-} \lim_{r \rightarrow 1} [I_{\mathcal{L}} \otimes F(rS'_1, \dots, rS'_m)](U \otimes I). \end{aligned}$$

Since the Hilbert space \mathcal{L} is separable, $\text{SOT-} \lim_{r \rightarrow 1} F(rS'_1, \dots, rS'_m) = \tilde{F}$, and $\|F(rS'_1, \dots, rS'_m)\| \leq \|F\|_\infty$, we also have

$$\text{SOT-} \lim_{r \rightarrow 1} I_{\mathcal{L}} \otimes F(rS'_1, \dots, rS'_m) = I_{\mathcal{L}} \otimes \tilde{F}.$$

Combining the result with relation (1.4), we conclude that

$$(1.5) \quad (U \otimes I)\widetilde{F \circ \Phi} = (I_{\mathcal{L}} \otimes \tilde{F})(U \otimes I).$$

Hence, we deduce that

$$\|F \circ \Phi\|_\infty = \|\widetilde{F \circ \Phi}\|_\infty = \|\tilde{F}\| = \|F\|_\infty.$$

Now, if we assume that F is inner, i.e., $\tilde{F}^* \tilde{F} = I$, then relation (1.5) implies $(\widetilde{F \circ \Phi})^* \widetilde{F \circ \Phi} = I$. Therefore, $F \circ \Phi$ is inner. Finally, assume that Φ is inner and F is outer, i.e., \tilde{F} has dense range. Using again relation (1.5), we deduce that $\widetilde{F \circ \Phi}$ has dense range and, therefore, $F \circ \Phi$ is outer. The proof is complete. \square

We recall a few well-known facts (see [52], [30], [57]) about fractional maps on the unit ball

$$[B(\mathcal{X}, \mathcal{Y})]_1^- := \{W \in B(\mathcal{X}, \mathcal{Y}) : \|W\| \leq 1\},$$

where \mathcal{X} and \mathcal{Y} are Hilbert spaces. We denote by $[B(\mathcal{X}, \mathcal{Y})]_1$ the open ball of strict contractions. Let $A, B \in [B(\mathcal{X}, \mathcal{Y})]_1^-$ be such that $\|A\| < 1$ and define $\Psi_A(B) \in B(\mathcal{X}, \mathcal{Y})$ by setting

$$(1.6) \quad \Psi_A(B) := A - D_{A^*}(I - BA^*)^{-1}BD_A,$$

where $D_A := (I - A^*A)^{1/2}$ and $D_{A^*} := (I - AA^*)^{1/2}$. One can show that, for any contractions $A, B, C \in B(\mathcal{X}, \mathcal{Y})$ with $\|A\| < 1$,

$$(1.7) \quad \begin{aligned} I - \Psi_A(B)\Psi_A(C)^* &= D_{A^*}(I - BA^*)^{-1}(I - BC^*)(I - AC^*)^{-1}D_{A^*}, \\ I - \Psi_A(B)^*\Psi_A(C) &= D_A(I - B^*A)^{-1}(I - B^*C)(I - A^*C)^{-1}D_A. \end{aligned}$$

Hence, we deduce that $\|\Psi_A(B)\| \leq 1$ and $\|\Psi_A(B)\| < 1$ when $\|B\| < 1$. Straightforward calculations reveal that

$$(1.8) \quad \Psi_A(0) = A, \quad \Psi_A(A) = 0, \quad \text{and} \quad \Psi_A(\Psi_A(B)) = B \quad \text{for any } B \in [B(\mathcal{X}, \mathcal{Y})]_1^-.$$

Consequently, the fractional map $\Psi_A : [B(\mathcal{X}, \mathcal{Y})]_1^- \rightarrow [B(\mathcal{X}, \mathcal{Y})]_1^-$ is a homeomorphism and, moreover, $\Psi_A([B(\mathcal{X}, \mathcal{Y})]_1) = [B(\mathcal{X}, \mathcal{Y})]_1$.

Consider the noncommutative Schur class

$$\mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G})) := \{G \in H_{\text{ball}}^\infty(B(\mathcal{E}, \mathcal{G})) : \|G\|_\infty \leq 1\},$$

which can be identified to the unit ball of the operator space $F_n^\infty \bar{\otimes} B(\mathcal{E}, \mathcal{G})$. We also use the notation

$$\mathcal{S}_{\text{ball}}^0(B(\mathcal{E}, \mathcal{G})) := \{G \in \mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G})) : G(0) = 0\}.$$

Fractional transforms of free holomorphic functions were considered in [43] (see the proof of Theorem 6.1). In what follows we expand on those ideas and provide new properties.

Theorem 1.3. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a bounded free holomorphic function with $\|F\|_\infty \leq 1$ and let \tilde{F} be its model boundary function. For each operator $A = I_{\mathcal{H}} \otimes A_0$ with $A_0 \in B(\mathcal{E}, \mathcal{G})$ and $\|A_0\| < 1$, we define the map*

$$\Psi_A[F] : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$$

by setting

$$\Psi_A[F] := A - D_{A^*}(I - FA^*)^{-1}FD_A.$$

Then the following statements hold:

- (i) $\Psi_A[F]$ is a bounded free holomorphic function with $\|\Psi_A[F]\|_\infty \leq 1$ and its boundary function has the following properties: $\widetilde{\Psi_A[F]} = \Psi_A(\tilde{F})$,

$$(1.9) \quad \begin{aligned} I - \widetilde{\Psi_A[F]} \widetilde{\Psi_A[F]}^* &= D_{A^*}(I - \tilde{F}A^*)^{-1}(I - \tilde{F}\tilde{F}^*)(I - A\tilde{F}^*)^{-1}D_{A^*}, \\ I - \widetilde{\Psi_A[F]}^* \widetilde{\Psi_A[F]} &= D_A(I - \tilde{F}^*A)^{-1}(I - \tilde{F}^*\tilde{F})(I - A^*\tilde{F})^{-1}D_A; \end{aligned}$$

- (ii) for any $X \in [B(\mathcal{H})^n]_1$,

$$\Psi_A[F](X) = (P_X \otimes I)\{\Psi_A(\tilde{F})\} = A - D_{A^*}[I - F(X)A^*]^{-1}F(X)D_A = \Psi_A[F(X)],$$

where P_X is the noncommutative Poisson at X ;

- (iii) $\Psi_A[0] = A$, $\Psi_A[A] = 0$, and $\Psi_A[\Psi_A[F]] = F$;
(iv) $\Psi_A : \mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G})) \rightarrow \mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G}))$ is a homeomorphism;
(v) \tilde{F} is inner if and only if $\widetilde{\Psi_A[F]}$ is inner;
(vi) \tilde{F} is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ if and only if $\widetilde{\Psi_A[F]}$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$.

Proof. To prove (i), note that FA^* is a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$ and $\|FA^*\|_\infty \leq \|A\| < 1$. Since the map $Y \mapsto (I - Y)^{-1}$ is a free holomorphic on $[B(\mathcal{K})]_1$, Theorem 1.1 implies that $(I - FA^*)^{-1}$ and, consequently, $\Psi_A[F]$ are bounded free holomorphic functions on $[B(\mathcal{H})^n]_1$. On the other hand, since

$$\|F\|_\infty = \sup_{r \in [0,1]} \|F(rS_1, \dots, rS_n)\| \leq 1$$

and using the properties of the fractional transform Ψ_A , we deduce that

$$\|\Psi_A[F](rS_1, \dots, rS_n)\| = \|\Psi_A(F(rS_1, \dots, rS_n))\| \leq 1$$

for any $r \in [0, 1)$. Hence $\|\Psi_A[F]\|_\infty \leq 1$. Since F is a bounded free holomorphic function, we know (see [39], [44]) that the boundary function

$$\tilde{F} := \text{SOT-}\lim_{r \rightarrow 1} F(rS_1, \dots, rS_n)$$

exists. Taking into account that $\|A\| < 1$ and $\|F(rS_1, \dots, rS_n)\| \leq \|\tilde{F}\| \leq 1$, one can easily see that

$$\text{SOT-}\lim_{r \rightarrow 1} (I - F(rS_1, \dots, rS_n)A^*)^{-1} = (I - \tilde{F}A^*)^{-1}$$

and, moreover,

$$\widetilde{\Psi_A[F]} = \text{SOT-}\lim_{r \rightarrow 1} \Psi_A(F(rS_1, \dots, rS_n)) = \Psi_A(\tilde{F}).$$

Now, notice that relation (1.9) follows from (1.7). This proves part (i).

Using the Poisson representation for bounded free holomorphic functions and the continuity of the Poisson transform in the operator norm topology, we obtain

$$\begin{aligned} \Psi_A[F](X) &= (P_X \otimes I) \{ \widetilde{\Psi_A[F]} \} = (P_X \otimes I) \{ \Psi_A[\tilde{F}] \} \\ &= A - D_{A^*} [I - F(X)A^*]^{-1} F(X) D_A = \Psi_A[F(X)] \end{aligned}$$

for any $X \in [B(\mathcal{H})^n]_1$, which proves part (ii). Hence and using relation (1.8), one can deduce (iii).

Now let us prove item (iv). Let $F, F_m \in \mathcal{S}_{\text{ball}}(B(\mathcal{E}, \mathcal{G}))$ and assume that $\|F_m - F\|_\infty \rightarrow 0$ as $m \rightarrow \infty$, which is equivalent to $\|\tilde{F}_m - \tilde{F}\| \rightarrow 0$. Using the fact that

$$\begin{aligned} \|(I - \tilde{F}_m^* A)^{-1} - (I - \tilde{F}^* A)^{-1}\| &\leq \|(I - \tilde{F}_m^* A)^{-1} (\tilde{F}_m A^* - \tilde{F} A^*) (I - \tilde{F}_m^* A)^{-1}\| \\ &\leq \frac{\|A\|}{(1 - \|A\|)^2} \|\tilde{F}_m - \tilde{F}\|, \end{aligned}$$

we deduce that $\Psi_A(\tilde{F}_m) \rightarrow \Psi_A(\tilde{F})$, as $m \rightarrow \infty$. Due to (i), we have

$$\lim_{m \rightarrow \infty} \|\Psi_A[F_m] - \Psi_A[F]\|_\infty = \lim_{m \rightarrow \infty} \|\Psi_A(\tilde{F}_m) - \Psi_A(\tilde{F})\| = 0.$$

Moreover, since $\Psi_A[\Psi_A[F]] = F$, we deduce that Ψ_A^{-1} is continuous, as well, in the uniform norm $\|\cdot\|_\infty$.

Note that item (v) follows from relation (1.9). To prove (vi), we assume that \tilde{F} is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$. Then, due to [39], $\tilde{F} = \lim_{r \rightarrow 1} F(rS_1, \dots, rS_n)$ in the operator norm topology. Since $\|F(rS_1, \dots, rS_n)A^*\| \leq \|F\|_\infty \|A\| < 1$, we deduce that

$$\lim_{r \rightarrow 1} (I - F(rS_1, \dots, rS_n)A^*)^{-1} = (I - \tilde{F}A^*)^{-1}$$

and, due to (iv),

$$\widetilde{\Psi_A[F]} = \lim_{r \rightarrow 1} \Psi_A[F](rS_1, \dots, rS_n) = \lim_{r \rightarrow 1} \Psi_A(F(rS_1, \dots, rS_n)) = \Psi_A(\tilde{F})$$

where the limits are in the operator norm topology. Using again [39], we conclude that $\widetilde{\Psi_A[F]}$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$. The converse follows using item (iv) and the fact that $\Psi_A[\Psi_A[F]] = F$. Indeed, if $\widetilde{\Psi_A[F]}$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$, then

$$\begin{aligned} \Psi_A\{\widetilde{\Psi_A[F]}\} &= \Psi_A\{\lim_{r \rightarrow 1} \Psi_A[F](rS_1, \dots, rS_n)\} \\ &= \lim_{r \rightarrow 1} \Psi_A\{\Psi_A(F(rS_1, \dots, rS_n))\} \\ &= \lim_{r \rightarrow 1} F(rS_1, \dots, rS_n) = \tilde{F}, \end{aligned}$$

where the limits are in the operator norm topology. Consequently, \tilde{F} is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$. The proof is complete. \square

Note that under the conditions of Theorem 1.3, we have

$$(1.10) \quad \begin{aligned} I - \Psi_A[F](X)\Psi_A[F](X)^* &= D_{A^*}[I - F(X)A^*]^{-1}[I - F(X)F(X)^*][I - AF(X)^*]^{-1}D_{A^*}, \\ I - \Psi_A[F](X)^*\Psi_A[F](X) &= D_A[I - F(X)^*A]^{-1}[I - F(X)^*F(X)][I - A^*F(X)]^{-1}D_A \end{aligned}$$

for any $X \in [B(\mathcal{H})^n]_1$. Moreover, we have

- (i) $\|F(X)\| < 1$ if and only if $\|\Psi_A[F](X)\| < 1$;
- (ii) if $X \in [B(\mathcal{H})^n]_1$, then $F(X)$ is an isometry (resp. co-isometry) if and only if $\Psi_A[F](X)$ has the same property.

We recall (see [39], [44]) that if $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is a free holomorphic function with coefficients in $B(\mathcal{E}, \mathcal{G})$ and $F_r(X) := F(rX)$ for any $X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_{1/r}$, $r \in (0, 1)$, then F_r is free holomorphic on $[B(\mathcal{H})^n]_{1/r}$ and

$$\|F_r\|_\infty = \sup_{\|X\| \leq r} \|F(X)\| = \sup_{\|X\| = r} \|F(X)\| = \|F(rS_1, \dots, rS_n)\|,$$

where S_1, \dots, S_n are the left creation operators. Moreover, the map $[0, 1) \ni r \mapsto \|F_r\|_\infty$ is increasing.

This result can be improved for free holomorphic functions with scalar coefficients. We recall that, in [45], we proved a maximum principle for free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_1$, with scalar coefficients. More precisely, we showed that if $f : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$ is a free holomorphic function and there exists $X_0 \in [B(\mathcal{H})^n]_1$ such that $\|f(X_0)\| \geq \|f(X)\|$ for any $X \in [B(\mathcal{H})^n]_1$, the f must be a constant. As a consequence of this principle and the noncommutative von Neumann inequality, one can easily obtain the following.

Proposition 1.4. *Let $f : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$ be a non-constant free holomorphic function with $\|f\|_\infty \leq 1$. Then the following statements hold:*

- (i) $\|f(X_1, \dots, X_n)\| < 1$ for any $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$;
- (ii) the map $[0, 1) \ni r \mapsto \|f_r\|_\infty$ is strictly increasing.

Proof. The first part follows immediately from the maximum principle for free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_1$. To prove the second part, let $0 \leq r_1 < r_2 < 1$. We recall that, if $r \in [0, 1)$, then the boundary function \tilde{f}_r is in the noncommutative disc algebra \mathcal{A}_n and $\|f_r\|_\infty = \|\tilde{f}_r\| = \|f_r(S_1, \dots, rS_n)\|$. Applying part (i) to f_{r_2} and $(X_1, \dots, X_n) := (\frac{r_1}{r_2}S_1, \dots, \frac{r_1}{r_2}S_n)$, we obtain

$$\|f_{r_1}\|_\infty = \|f_{r_1}(S_1, \dots, S_n)\| = \left\| f_{r_2} \left(\frac{r_1}{r_2}S_1, \dots, \frac{r_1}{r_2}S_n \right) \right\| < \|f_{r_2}(S_1, \dots, S_n)\| = \|f_{r_2}\|_\infty,$$

which completes the proof. \square

Now, using fractional transforms, and the noncommutative version of Schwarz's lemma [39], we extend the maximum principle to free holomorphic functions with operator-valued coefficients.

Theorem 1.5. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a bounded free holomorphic function with $\|F(0)\| < \|F\|_\infty$. Then there is no $X_0 \in [B(\mathcal{H})^n]_1$ such that*

$$\|F(X_0)\| = \|F\|_\infty.$$

Proof. Without loss of generality, we can assume that $\|F\|_\infty = 1$. Set $A := F(0)$ and let $G := \Psi_A[F]$. Due to Theorem 1.3, G is a bounded free holomorphic function with $\|G\|_\infty \leq 1$ and $G(0) = \Psi_A(A) = 0$. Applying the noncommutative Schwarz lemma (see [39]), we obtain

$$\|G(X)\| = \|\Psi_A(F(X))\| \leq \|X\| < 1, \quad X \in [B(\mathcal{H})^n]_1.$$

Using again Theorem 1.3, we have $(\Psi_A \circ \Psi_A)[F] = F$ and, therefore,

$$\|F(X)\| < 1 = \|F\|_\infty, \quad X \in [B(\mathcal{H})^n]_1.$$

The proof is complete. \square

We need to make a few remarks, which are familiar in the case $n = 1$ (see [54]). First, we recall (see [45]) that in the scalar case, $\mathcal{E} = \mathcal{G} = \mathbb{C}$, if $f : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$ is a free holomorphic function and $\|f\|_\infty = |f(0)|$, then f must be a constant. On the other hand, if F is not a scalar free holomorphic function and $\|F\|_\infty = \|F(0)\|$, then Theorem 1.5 fails. Indeed, take $\mathcal{E} = \mathcal{G} = \mathbb{C}^2$, and

$$F(X) = \begin{bmatrix} I & 0 \\ 0 & g(X) \end{bmatrix},$$

where g is a scalar free holomorphic function with $\|g(X)\| < 1$ for $X = (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$ (for example, $g(X) = X_\alpha$, $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geq 1$). Note that $\|F(X)\| = 1 = \|F(0)\|$ for any $X \in [B(\mathcal{H})^n]_1$.

We also mention that, if $\|F\|_\infty = 1$ and $F(0)$ is an isometry, then F must be a constant. Indeed, if F has the representation $f(X) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)}$, then, due to [40], we have

$$\sum_{|\alpha|=k} A_{(\alpha)}^* A_{(\alpha)} \leq I - F(0)^* F(0) \quad \text{for } k = 1, 2, \dots$$

Hence, we deduce our assertion.

Using Theorem 1.5, one can prove the following result. Since the proof is similar to that of Proposition 1.4, we shall omit it.

Corollary 1.6. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a bounded free holomorphic function with $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$. Then*

$$\|F(X)\| < \|F\|_\infty \quad \text{for any } X \in [B(\mathcal{H})^n]_1.$$

If $F(0) = 0$, then the map $[0, 1) \ni r \mapsto \|F_r\|_\infty$ is strictly increasing.

We remark that, in general, under the conditions of Corollary 1.6, but without the condition $F(0) = 0$, the map $[0, 1) \ni r \mapsto \|F_r\|_\infty$ is not necessarily strictly increasing. Indeed, take

$$F(X_1, \dots, X_n) = \begin{bmatrix} \frac{1}{3}I & 0 \\ 0 & \frac{1}{2}X_1 \end{bmatrix}$$

and note that $\|F\|_\infty = \frac{1}{2}$ and

$$\|F(rS_1, \dots, rS_n)\| = \begin{cases} \frac{1}{3} & r \in [0, \frac{2}{3}] \\ \frac{r}{2} & r \in (\frac{2}{3}, 1]. \end{cases}$$

Denote by $H_{\mathbf{ball}}^+(B(\mathcal{E}))$ the set of all free holomorphic functions f on the noncommutative ball $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$, where \mathcal{E} is a separable Hilbert space, such that $\Re f \geq 0$, where

$$(\Re F)(X) := \frac{F(X)^* + F(X)}{2}, \quad X \in [B(\mathcal{H})^n]_1.$$

Let $[H_{\mathbf{ball}}^\infty(B(\mathcal{E}))]^{\text{inv}}$ denote the set of all bounded free holomorphic functions on $[B(\mathcal{H})^n]_1$ with representation $F(X_1, \dots, X_n) = \sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)} \otimes X_\alpha$ such that $I_{\mathcal{E}} - A_{(0)}$ is an invertible operator in $B(\mathcal{E})$. According to [41], the noncommutative Cayley transform defined by

$$\mathcal{C}[F] := [F - 1][1 + F]^{-1}$$

is a bijection between $H_{\mathbf{ball}}^+(B(\mathcal{E}))$ and the unit ball of $[H_{\mathbf{ball}}^\infty(B(\mathcal{E}))]^{\text{inv}}$. In this case, we have

$$\mathcal{C}^{-1}[G] = [I + G][I - G]^{-1}.$$

Consider also the set

$$\mathbf{H}_1^+(B(\mathcal{E})) := \{f \in H_{\mathbf{ball}}^+(B(\mathcal{E})) : f(0) = I\}.$$

Now, we recall that the restriction to $\mathbf{H}_1^+(B(\mathcal{E}))$ of the *noncommutative Cayley transform* is a bijection $\mathcal{C} : \mathbf{H}_1^+(B(\mathcal{E})) \rightarrow \mathcal{S}_{\mathbf{ball}}^0(B(\mathcal{E}))$, where the noncommutative Schur class $\mathcal{S}_{\mathbf{ball}}^0(B(\mathcal{E}))$ was introduced before Theorem 1.3.

Using fractional transforms, we can prove the following theorem concerning the structure of bounded free holomorphic functions.

Theorem 1.7. *A map $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ is a bounded free holomorphic function such that $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$, if and only if there exist a strict contraction $A_0 \in B(\mathcal{E})$, an n -tuple of isometries (V_1, \dots, V_n) on a Hilbert space \mathcal{K} , with orthogonal ranges, and an isometry $W : \mathcal{E} \rightarrow \mathcal{K}$, such that*

$$F = (\Psi_{I \otimes A_0} \circ \mathcal{C})[G],$$

where \mathcal{C} is the noncommutative Cayley transform and G is defined by

$$G(X_1, \dots, X_n) = (I \otimes W^*) [2(I - X_1 \otimes V_1^* - \dots - X_n \otimes V_n^*)^{-1} - I] (I \otimes W)$$

for any $X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$. In this case, $F(0) = I \otimes A_0$.

Proof. Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ be a bounded free holomorphic function with $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$. Then $F \in \mathcal{S}_{\text{ball}}(B(\mathcal{E}))$ and, due to Theorem 1.3, $\Psi_{F(0)}[F] \in \mathcal{S}_{\text{ball}}^0(B(\mathcal{E}))$. Since the noncommutative Cayley transform $\mathcal{C} : \mathbf{H}_1^+(B(\mathcal{E})) \rightarrow \mathcal{S}_{\text{ball}}^0(B(\mathcal{E}))$ is a bijection, we deduce that $\mathcal{C}^{-1}(\Psi_{F(0)}[F]) \in \mathbf{H}_1^+(B(\mathcal{E}))$.

According to [44], a free holomorphic function G is in $\mathbf{H}_1^+(B(\mathcal{E}))$, i.e., $G(0) = I$ and $\Re G \geq 0$, if and only if there exists an n -tuple of isometries (V_1, \dots, V_n) on a Hilbert space \mathcal{K} , with orthogonal ranges, and an isometry $W : \mathcal{E} \rightarrow \mathcal{K}$ such that

$$G(X_1, \dots, X_n) = (I \otimes W^*) [2(I - X_1 \otimes V_1^* - \dots - X_n \otimes V_n^*)^{-1} - I] (I \otimes W).$$

This completes the proof. \square

We remark that, in the scalar case, i.e., $\mathcal{E} = \mathbb{C}$, due to the maximum principle for free holomorphic functions, any nonconstant free holomorphic function f such that $\|f\|_\infty \leq 1$, has the property that $|f(0)| < 1$. Therefore, we can apply Theorem 1.7 and obtain a characterization and a parametrization of all bounded free holomorphic functions on $[B(\mathcal{H})^n]_1$.

2. VITALI CONVERGENCE AND IDENTITY THEOREM FOR FREE HOLOMORPHIC FUNCTIONS

In this section, we provide a Vitali type convergence theorem for uniformly bounded sequences of free holomorphic functions with operator-valued coefficients.

Theorem 2.1. *Let $\{F_m\}_{m=1}^\infty$ be a uniformly bounded sequence of free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$. Let $\{A^{(k)}\}_{k=1}^\infty \subset [B(\mathcal{H})^n]_1$ be a sequence of operators with the following properties:*

- (i) $A^{(k)}$ is bounded below, for each $k = 1, 2, \dots$;
- (ii) $\lim_{k \rightarrow \infty} \|A^{(k)}\| = 0$;
- (iii) $\lim_{m \rightarrow \infty} F_m(A^{(k)})$ exists in the operator norm topology, for each $k = 1, 2, \dots$.

Then there exists a free holomorphic function F on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ such that F_m converges to F uniformly on any closed ball $[B(\mathcal{H})^n]_r^-$, $r \in [0, 1)$.

Proof. For each $m = 1, 2, \dots$, let F_m have the representation

$$F_m(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes C_{(\alpha)}^{(m)},$$

where the series converges in the operator norm topology for any $X = (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$. Let $M > 0$ be such that $\|F_m(X)\| \leq M$ for any $X \in [B(\mathcal{H})^n]_1$ and $m = 1, 2, \dots$. Due to the Cauchy type estimation of Theorem 2.1 from [39], we have

$$(2.1) \quad \left\| \sum_{|\alpha|=j} (C_{(\alpha)}^{(m)})^* C_{(\alpha)}^{(m)} \right\|^{1/2} \leq M, \quad \text{for any } m = 1, 2, \dots, \text{ and } j = 0, 1, \dots$$

Since $\|F_m(X) - F_m(0)\| \leq 2M$ for $X \in [B(\mathcal{H})^n]_1$, the Schwarz type result for free holomorphic functions [39] implies

$$\|F_m(A^{(k)}) - I_{\mathcal{H}} \otimes A_{(0)}^{(m)}\| \leq 2M\|A^{(k)}\|$$

for any $m, k = 1, 2, \dots$. Hence, we deduce that

$$\begin{aligned} \|A_{(0)}^{(m)} - A_{(0)}^{(q)}\| &\leq \|I_{\mathcal{H}} \otimes A_{(0)}^{(m)} - F_m(A^{(k)})\| + \|F_m(A^{(k)}) - F_q(A^{(k)})\| + \|F_q(A^{(k)}) - I_{\mathcal{H}} \otimes A_{(0)}^{(m)}\| \\ &\leq 4M\|A^{(k)}\| + \|F_m(A^{(k)}) - F_q(A^{(k)})\|. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} \|A^{(k)}\| = 0$ and $\lim_{m \rightarrow \infty} F_m(A^{(k)})$ exists in the operator norm topology, for each $k = 1, 2, \dots$, we deduce that $C_{(0)} := \lim_{m \rightarrow \infty} C_{(0)}^{(m)}$ exists.

Let $F_{m,0} := F_m$ and note that

$$F_{m,0} = I \otimes C_{(0)}^{(m)} + [X_1 \otimes I_{\mathcal{E}}, \dots, X_n \otimes I_{\mathcal{E}}] F_{m,1}(X)$$

where

$$F_{m,1}(X) = \begin{bmatrix} F_{m,1}^{(g_1)}(X) \\ \vdots \\ F_{m,1}^{(g_n)}(X) \end{bmatrix}$$

and

$$F_{m,1}^{(g_i)}(X) = I_{\mathcal{H}} \otimes C_{(g_i)}^{(m)} + \sum_{j=1} \sum_{|\beta|=j} X_{\beta} \otimes C_{(g_i\beta)}^{(m)}, \quad i = 1, \dots, n.$$

By induction over $q = 0, 1, 2, \dots$, we can easily prove that

$$\sum_{|\alpha| \geq q} X_{\alpha} \otimes C_{(\alpha)} = [X_{\beta} \otimes I_{\mathcal{E}} : |\beta| = q] F_{m,q}(X),$$

where

$$F_{m,q}(X) := \begin{bmatrix} F_{m,q}^{(\beta)}(X) \\ \vdots \\ |\beta| = q \end{bmatrix}$$

and

$$F_{m,q}^{(\beta)}(X) := I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} + \sum_{j=1} \sum_{|\gamma|=j} X_{\gamma} \otimes C_{(\beta\gamma)}^{(m)} \quad \text{for } |\beta| = q.$$

Now, note that, for each $i = 1, \dots, n$, we have

$$F_{m,1}^{(g_i)}(X) = I_{\mathcal{H}} \otimes C_{(g_i)}^{(m)} + [X_1 \otimes I_{\mathcal{E}}, \dots, X_n \otimes I_{\mathcal{E}}] \begin{bmatrix} F_{m,2}^{(g_i g_1)}(X) \\ \vdots \\ F_{m,2}^{(g_i g_n)}(X) \end{bmatrix}.$$

Consequently, we have

$$(2.2) \quad F_{m,1}(X) = \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(g_1)}^{(m)} \\ \vdots \\ I_{\mathcal{H}} \otimes C_{(g_n)}^{(m)} \end{bmatrix} + ([X_1 \otimes I_{\mathcal{E}}, \dots, X_n \otimes I_{\mathcal{E}}] \otimes I_{\mathbb{C}^{N_1}}) F_{m,2}(X),$$

where $N_1 := n$. One can easily prove by induction over $q = 0, 1, \dots$ that

$$(2.3) \quad F_{m,q}(X) = \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q \end{bmatrix} + ([X_1 \otimes I_{\mathcal{E}}, \dots, X_n \otimes I_{\mathcal{E}}] \otimes I_{\mathbb{C}^{N_q}}) F_{m,q+1}(X)$$

for any $X \in [B(\mathcal{H})^n]_1$ and $m = 1, 2, \dots$, where $N_q := \text{card} \{\beta \in \mathbb{F}_n^+ : |\beta| = q\}$.

In what follows we prove by induction over $p = 0, 1, \dots$ the following statements:

- (a) $\lim_{m \rightarrow \infty} F_{m,p}(A^{(k)})$ exists in the operator norm topology, for each $k = 1, 2, \dots$;
- (b) $\|F_{m,p}(X)\| \leq (p+1)M$ for any $X \in [B(\mathcal{H})^n]_1$ and $m = 1, 2, \dots$;
- (c) $\left\| F_{m,p}(A^{(k)}) - \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = p \end{bmatrix} \right\| \leq (p+2)M\|A^{(k)}\|$ for any $k, m = 1, 2, \dots$;
- (d) $\begin{bmatrix} C_{(\beta)} \\ \vdots \\ |\beta| = p \end{bmatrix} := \lim_{m \rightarrow \infty} \begin{bmatrix} C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = p \end{bmatrix}$ exists in the operator norm topology.

Assume that these relations hold for $p = q$. Using relation (2.3) when $X = A^{(k)}$ and taking into account (a) and (d) (when $p = q$), we deduce that the sequence $\{(A^{(k)} \otimes I_{\mathcal{E} \otimes \mathbb{C}^{N_{q+1}}})F_{m,q+1}(A^{(k)})\}_{m=1}^{\infty}$ is convergent in the operator norm topology and, consequently, a Cauchy sequence. On the other hand, since $A^{(k)}$ is bounded below, there exists $C > 0$ such that $\|A^{(k)}y\| \geq C\|y\|$ for any $y \in \oplus_{i=1}^n \mathcal{H}$. This implies that

$$\begin{aligned} & \left\| (A^{(k)} \otimes I_{\mathcal{E} \otimes \mathbb{C}^{N_{q+1}}})F_{m,q+1}(A^{(k)})x - (A^{(k)} \otimes I_{\mathcal{E} \otimes \mathbb{C}^{N_{q+1}}})F_{t,q+1}(A^{(k)})x \right\| \\ & \geq C \left\| F_{m,q+1}(A^{(k)})x - F_{t,q+1}(A^{(k)})x \right\| \end{aligned}$$

for any $x \in \mathcal{H} \otimes \mathcal{E} \otimes \mathbb{C}^{N_{q+1}}$ and $m, t = 1, 2, \dots$. Hence, we deduce that $\{F_{m,q+1}(A^{(k)})\}_{m=1}^{\infty}$ is a Cauchy sequence and, therefore, $\lim_{m \rightarrow \infty} F_{m,q+1}(A^{(k)})$ exists.

Now, due to relation (2.1) and (b), we have

$$\left\| F_{m,q}(X) - \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q \end{bmatrix} \right\| \leq (q+2)M, \quad X \in [B(\mathcal{H})^n]_1.$$

Using relation (2.2) and the noncommutative Schwarz lemma, we obtain

$$\|(X \otimes I_{\mathcal{E} \otimes \mathbb{C}^{N_q}})F_{m,q+1}(X)\| = \left\| F_{m,q}(X) - \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q \end{bmatrix} \right\| \leq (q+2)M\|X\|$$

for any $X \in [B(\mathcal{H})^n]_1$, which implies

$$\|F_{m,q+1}(X)\| \leq (q+2)M.$$

Hence and using again (2.1), we obtain

$$\left\| F_{m,q+1}(X) - \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q+1 \end{bmatrix} \right\| \leq (q+3)M$$

for any $X \in [B(\mathcal{H})^n]_1$. Once again, applying the Schwarz lemma for free holomorphic functions, we deduce that

$$\left\| F_{m,q+1}(A^{(k)}) - \begin{bmatrix} I_{\mathcal{H}} \otimes C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q+1 \end{bmatrix} \right\| \leq (q+3)M\|A^{(k)}\|$$

for any $k, m = 1, 2, \dots$, which is condition (c), when $p = q + 1$. Now, note that

$$\begin{aligned} \left\| \begin{bmatrix} C_{(\beta)}^{(s)} - C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q + 1 \end{bmatrix} \right\| &\leq \left\| \begin{bmatrix} C_{(\beta)}^{(s)} \\ \vdots \\ |\beta| = q + 1 \end{bmatrix} - F_{s,q+1}(A^{(k)}) \right\| + \left\| F_{s,q+1}(A^{(k)}) - F_{m,q+1}(A^{(k)}) \right\| \\ &\quad + \left\| F_{m,q+1}(A^{(k)}) - \begin{bmatrix} C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q + 1 \end{bmatrix} \right\| \\ &\leq 2(q+3)M\|A^{(k)}\| + \left\| F_{s,q+1}(A^{(k)}) - F_{m,q+1}(A^{(k)}) \right\|. \end{aligned}$$

for any $k, s, m = 1, 2, \dots$. Since $\{F_{m,q+1}(A^{(k)})\}_{m=1}^\infty$ is a Cauchy sequence and $\lim_{k \rightarrow \infty} \|A^{(k)}\| = 0$, we deduce condition (d) when $p = q + 1$, i.e.,

$$\begin{bmatrix} C_{(\beta)} \\ \vdots \\ |\beta| = q + 1 \end{bmatrix} := \lim_{m \rightarrow \infty} \begin{bmatrix} C_{(\beta)}^{(m)} \\ \vdots \\ |\beta| = q + 1 \end{bmatrix}$$

exists in the operator norm topology. This concludes our proof by induction.

Now, due to (2.1), we deduce that

$$\left\| \sum_{|\alpha|=j} C_{(\alpha)}^* C_{(\alpha)} \right\|^{1/2} \leq M, \quad \text{for any } j = 0, 1, \dots,$$

which implies $\limsup_{k \rightarrow \infty} \left\| \sum_{|\alpha|=k} C_{(\alpha)}^* C_{(\alpha)} \right\|^{\frac{1}{2k}} \leq 1$. Consequently, the mapping $F(X) := \sum_{j=0}^\infty \sum_{|\alpha|=j} X_\alpha \otimes C_{(\alpha)}$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$. If $\|X\| \leq r < 1$, then we have

$$\begin{aligned} \|F_m(X) - F(X)\| &\leq \sum_{j=0}^{p-1} \left\| \sum_{|\alpha|=j} X_\alpha \otimes (C_{(\alpha)}^{(m)} - C_{(\alpha)}) \right\| + \sum_{j=p}^\infty \left\| \sum_{|\alpha|=j} X_\alpha \otimes (C_{(\alpha)}^{(m)} - C_{(\alpha)}) \right\| \\ &\leq \sum_{j=0}^{p-1} \left\| \begin{bmatrix} C_{(\alpha)}^{(m)} - C_{(\alpha)} \\ \vdots \\ |\alpha| = j \end{bmatrix} \right\| + \sum_{j=p}^\infty r^j \left\| \begin{bmatrix} C_{(\alpha)}^{(m)} - C_{(\alpha)} \\ \vdots \\ |\alpha| = j \end{bmatrix} \right\| \\ &\leq \sum_{j=0}^{p-1} \left\| \begin{bmatrix} C_{(\alpha)}^{(m)} - C_{(\alpha)} \\ \vdots \\ |\alpha| = j \end{bmatrix} \right\| + 2M \frac{r^p}{1-r}. \end{aligned}$$

Hence and due to relation (d), we deduce that $\|F_m(X) - F(X)\| \rightarrow 0$, as $m \rightarrow \infty$, uniformly for $X \in [B(\mathcal{H})^n]_r^-$, $r \in [0, 1)$. The proof is complete. \square

Corollary 2.2. *Let F, G be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$. If there exists a sequence $\{A^{(k)}\}_{k=1}^\infty \subset [B(\mathcal{H})^n]_1$ of bounded below operators such that $\lim_{k \rightarrow \infty} \|A^{(k)}\| = 0$ and $F(A^{(k)}) = G(A^{(k)})$ for any $k = 1, 2, \dots$, then $F = G$.*

Proof. When F and G are bounded on $[B(\mathcal{H})^n]_1$, we can apply Theorem 2.1 to the sequence F, G, F, G, \dots , and deduce that $F = G$. Otherwise, let $r \in (0, 1)$ be such that $\|A^{(k)}\| < r$ and consider $F_r(X) := F(rX)$ and $G_r(X) := G(rX)$ for $X \in [B(\mathcal{H})^n]_1$. Since F_r and G_r are bounded free holomorphic functions on $[B(\mathcal{H})^n]_1$ and

$$F_r(r^{-1}A^{(k)}) = F(A^{(k)}) = G(A^{(k)}) = G_r(r^{-1}A^{(k)})$$

we can apply the first part of the proof and deduce that $F_r = G_r$. Consequently, $F(rS_1, \dots, rS_n) = G(rS_1, \dots, rS_n)$, where S_1, \dots, S_n are the left creation operators. Hence $F = G$. \square

Remark 2.3. *Theorem 2.1 fails if the operators $A^{(k)}$ are not bounded below.*

Proof. Let $m = 2, 3, \dots$, and consider the sequence of strict row contractions

$$A^{(k)} := \left[\frac{1}{k} P_{\mathcal{P}_{m-1}} S_1|_{\mathcal{P}_{m-1}}, \dots, \frac{1}{k} P_{\mathcal{P}_{m-1}} S_n|_{\mathcal{P}_{m-1}} \right], \quad k = 1, 2, \dots,$$

where S_1, \dots, S_n are the left creation operators and \mathcal{P}_{m-1} is the subspace of $F^2(H_n)$ spanned by the vectors e_α , with $\alpha \in \mathbb{F}_n^+$ and $|\alpha| \leq m-1$. Let F and G be any free holomorphic functions on $[B(\mathcal{H})^n]_1$ such that $F(X) - G(X) = X_\beta$ for some $\beta \in \mathbb{F}_n^+$ with $|\beta| = m$. Since $A_\alpha^{(k)} = 0$ for $|\alpha| \geq m$, we have

$$F(A^{(k)}) = G(A^{(k)}), \quad k \geq 2,$$

and $\lim_{k \rightarrow \infty} \|A^{(k)}\| = 0$. However, $F \neq G$. \square

We should mention that in the particular case when $n = 1$, $\mathcal{E} = \mathbb{C}$, and $\{A^{(k)}\}$ is a sequence of invertible strict contractions, we recover the corresponding results obtained by Fan [14].

3. FREE HOLOMORPHIC FUNCTIONS WITH THE RADIAL INFIMUM PROPERTY

We introduce the class of free holomorphic functions with the radial infimum property, obtain several characterizations, and consider several examples. We study the radial infimum property in connection with products, direct sums, and compositions of free holomorphic functions. We also show that the class of functions with the radial infimum property is invariant under the fractional transforms of Section 1. These results are important in the following sections.

Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$. Due to [39] and [44], the model boundary function

$$\tilde{F}(S_1, \dots, S_n) := \text{SOT-}\lim_{r \rightarrow 1} F(rS_1, \dots, rS_n)$$

exists, and F has the *radial supremum property*, i.e.,

$$\lim_{r \rightarrow 1} \sup_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\| = \|F\|_\infty.$$

We introduce now the class of free holomorphic functions with the radial infimum property. We say that F has the *radial infimum property* if

$$\liminf_{r \rightarrow 1} \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\| = \|F\|_\infty.$$

Proposition 3.1. *If $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is a bounded free holomorphic function with the radial infimum property such that $\|F\|_\infty = 1$, then F is inner.*

Proof. We have to show that the boundary function $\tilde{F}(S_1, \dots, S_n)$ is an isometry. To this end, denote

$$\mu(r) := \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\| \quad \text{for } r \in [0, 1].$$

Due to the noncommutative von Neumann inequality, we have

$$\mu(r) \leq \frac{\|F(rS_1, \dots, rS_n)y\|}{\|y\|} \leq \|F(rS_1, \dots, rS_n)\| \leq \|F\|_\infty$$

for any $y \in F^2(H_n) \otimes \mathcal{E}$, $y \neq 0$ and $r \in [0, 1]$. Taking into account that $\liminf_{r \rightarrow 1} \mu(r) = \|F\|_\infty$, we deduce that $\lim_{r \rightarrow 1} \|F(rS_1, \dots, rS_n)y\| = \|y\|$. Since $\tilde{F}(S_1, \dots, S_n) := \text{SOT-}\lim_{r \rightarrow 1} F(rS_1, \dots, rS_n)$, it is clear that $\|\tilde{F}(S_1, \dots, S_n)y\| = \|y\|$ for any $y \in F^2(H_n) \otimes \mathcal{E}$, which Shows that F is inner and completes the proof. \square

Now, we present several characterizations for free holomorphic functions with the radial infimum property.

Theorem 3.2. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a bounded free holomorphic function with $\|F\|_\infty = 1$. Then the following statements are equivalent.*

- (i) *F has the radial infimum property.*

- (ii) $\lim_{r \rightarrow 1} \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\| = 1$.
- (iii) For every $\epsilon \in (0, 1)$ there is $\delta \in (0, 1)$ such that

$$F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \geq (1 - \epsilon)I \quad \text{for any } r \in (\delta, 1).$$
- (iv) There exist constants $c(r) \in (0, 1]$, $r \in (0, 1)$, with $\lim_{r \rightarrow 1} c(r) = 1$ such that

$$F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \geq c(r)I.$$
- (v) There is $\delta \in (0, 1)$ such that $F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)$ is invertible for any $r \in (\delta, 1)$ and

$$\lim_{r \rightarrow 1} \left\| [F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)]^{-1} \right\| = 1.$$

Proof. The equivalence of (i) with (ii) is clear if one takes into account the inequality

$$\|F(rS_1, \dots, rS_n)\| \leq \|F\|_\infty, \quad r \in [0, 1).$$

Since the equivalence of (ii) with (iii) is straightforward, we leave it to the reader. To prove the implication (ii) \implies (iv), define

$$\mu(r) := \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\|, \quad r \in [0, 1),$$

and note that $0 \leq \mu(r) \leq \|F(rS_1, \dots, rS_n)\| \leq \|F\|_\infty = 1$ and

$$\|F(rS_1, \dots, rS_n)x\| \geq \mu(r)\|x\| \quad \text{for any } x \in F^2(H_n) \otimes \mathcal{E}.$$

Since the latter inequality is equivalent to

$$F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \geq \mu(r)^2 I,$$

if (ii) holds, then $\lim_{r \rightarrow 1} \mu(r) = 1$ and (iv) follows. Conversely, assume that (iv) holds. Then we have

$$\|F(rS_1, \dots, rS_n)x\|^2 \geq c(r)\|x\|^2 \quad \text{for any } x \in F^2(H_n) \otimes \mathcal{E},$$

which implies

$$c(r)^{1/2} \leq \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)x\| \leq \|F(rS_1, \dots, rS_n)\| \leq \|F\|_\infty = 1.$$

Since $\lim_{r \rightarrow 1} c(r) = 1$, we deduce item (ii).

It remains to prove that (iv) \leftrightarrow (v). First, assume that condition (iv) holds. Note that the inequality

$$(3.1) \quad F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \geq c(r)I$$

is equivalent to

$$(3.2) \quad \|F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)x\| \geq c(r)\|x\| \quad \text{for any } x \in F^2(H_n) \otimes \mathcal{E}.$$

Indeed, if (3.1) holds, then

$$\|F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)x\| \geq \langle F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)x, x \rangle \geq c(r)\|x\|^2,$$

which proves one implication. Conversely, if (3.2) holds, then, by squaring, we deduce that

$$[F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)]^2 \geq c(r)^2 I.$$

Hence, we obtain relation (3.1). Now, denote

$$(3.3) \quad d(r) := \inf_{\|x\|=1} \|F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)x\|, \quad r \in (0, 1),$$

and note that $0 < c(r) \leq d(r) \leq 1$. Hence, using (3.2) and condition (iv), we deduce that the positive operator $F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)$ is invertible and

$$\frac{1}{c(r)} \geq \|[F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)]^{-1}\| = \frac{1}{d(r)} \geq 1.$$

Since $\lim_{r \rightarrow 1} c(r) = 1$, we obtain item (v). Conversely, assume now that condition (v) holds. Since $\|[F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)]^{-1}\| = \frac{1}{d(r)}$, where $d(r)$ is given by (3.3), we have $\lim_{r \rightarrow 1} d(r) = 1$. On the other hand, due to (3.3), we also have

$$\|F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)x\| \geq d(r)\|x\| \quad \text{for any } x \in F^2(H_n) \otimes \mathcal{E},$$

which, as proved above, is equivalent to $F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \geq d(r)I$. Since $\lim_{r \rightarrow 1} d(r) = 1$, we deduce (iv) and complete the proof. \square

Now we consider several examples of bounded free holomorphic functions with the radial infimum property. Another notation is necessary. If $\omega, \gamma \in \mathbb{F}_n^+$, we say that $\omega \geq \gamma$ if there is $\sigma \in \mathbb{F}_n^+$ such that $\omega = \gamma\sigma$.

Example 3.3. Let $A_{(\alpha)} \in B(\mathcal{E}, \mathcal{G})$ and let F, G, φ be free holomorphic function on $[B(\mathcal{H})^n]_1$ having the following forms:

- (i) $F(X_1, \dots, X_n) = \sum_{|\alpha|=m} X_\alpha \otimes A_{(\alpha)}$, where $\sum_{|\alpha|=m} A_\alpha^* A_\alpha = I_{\mathcal{E}}$ and $m \in \mathbb{N}$;
- (ii) $G(X_1, \dots, X_n) = \sum_{k=1}^n \sum_{|\beta|=k, \beta \geq g_k} X_\beta \otimes A_{(\beta)}$, where $\sum_{k=1}^n \sum_{|\beta|=k, \beta \geq g_k} A_{(\beta)}^* A_{(\beta)} = I$ and g_1, \dots, g_n are the generators of the free semigroup \mathbb{F}_n^+ ;
- (iii) $\varphi(X_1, \dots, X_n) = \sum_{k=0}^\infty a_k X_2^k X_1$, where $a_k \in \mathbb{C}$ with $\sum_{k=0}^\infty |a_k|^2 = 1$.

Then, F, G, φ have the radial infimum property.

Proof. Since S_1, \dots, S_n satisfy the relation $S_j^* S_i = \delta_{ij} I$ for $i, j = 1, \dots, n$, one can easily see that $\{S_\alpha\}_{|\alpha|=m}$ is a sequence of isometries with orthogonal ranges. Consequently, we have

$$F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) = r^m I.$$

Applying Theorem 3.2, we deduce that F has the radial infimum property. Similarly, one can prove that

$$\begin{aligned} G(rS_1, \dots, rS_n)^* G(rS_1, \dots, rS_n) &= \sum_{k=1}^n \sum_{|\beta|=k, \beta \geq g_k} r^{2|\beta|} A_{(\beta)}^* A_{(\beta)} \\ &\geq r^{2n} \sum_{k=1}^n \sum_{|\beta|=k, \beta \geq g_k} A_{(\beta)}^* A_{(\beta)} = r^{2n} I. \end{aligned}$$

Consequently, G has the radial infimum property. Finally, note that $\{S_2^k S_1\}_{k=0}^\infty$ is a sequence of isometries with orthogonal ranges and, consequently

$$\varphi(rS_1, \dots, rS_n)^* \varphi(rS_1, \dots, rS_n) = r^2 \sum_{k=0}^\infty r^{2k} |a_k|^2 \leq \sum_{k=1}^\infty |a_k|^2 = 1.$$

Hence φ is a bounded free holomorphic function and, taking into account that $\lim_{r \rightarrow 1} r^2 \sum_{k=0}^\infty r^{2k} |a_k|^2 = 1$, Theorem 3.2 shows that φ has the radial infimum property. \square

Proposition 3.4. If F, G are bounded free holomorphic functions with the radial infimum property, then so is their product FG . If, in addition, $\|F\|_\infty = \|G\|_\infty$, then $\begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix}$ has the radial infimum property.

Proof. Without loss of generality, one can assume that $\|F\|_\infty = \|G\|_\infty = 1$. According to Theorem 3.2, there exist constants $c(r), d(r) \in (0, 1]$, $r \in (0, 1)$, with $\lim_{r \rightarrow 1} c(r) = \lim_{r \rightarrow 1} d(r) = 1$ such that

$$F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \geq c(r)I \quad \text{and} \quad G(rS_1, \dots, rS_n)^* G(rS_1, \dots, rS_n) \geq d(r)I.$$

Hence, we deduce that

$$G(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) G(rS_1, \dots, rS_n) \geq c(r)d(r)I.$$

Applying again Theorem 3.2, we conclude that the product FG has the radial infimum property. To prove the second part of this proposition, note that

$$\begin{bmatrix} F(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) & 0 \\ 0 & G(rS_1, \dots, rS_n)^* G(rS_1, \dots, rS_n) \end{bmatrix} \geq \min\{c(r), d(r)\}I$$

and $\lim_{r \rightarrow 1} \min\{c(r), d(r)\} = 1$. Applying again Theorem 3.2, we complete the proof. \square

The next result will provide several classes of free holomorphic functions with the radial infimum property.

Theorem 3.5. *Let $F : [B(\mathcal{H})^m]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a bounded free holomorphic function, and let $\varphi : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1$ be an inner free holomorphic function. Then the following statements hold.*

- (i) *If F is inner and $\tilde{F} \in \mathcal{A}_m \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$, then F has the radial infimum property.*
- (ii) *If F is inner, $\tilde{F} \in \mathcal{A}_m \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$, and $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_m)$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathbb{C}^m, \mathbb{C})$ with $\tilde{\varphi}_1$ non-unitary if $m = 1$, then the composition $F \circ \varphi$ has the radial infimum property.*
- (iii) *If F has the radial infimum property and φ is homogeneous of degree $q \geq 1$, then $F \circ \varphi$ has the radial infimum property.*
- (iv) *If $A := I \otimes A_0$, $A_0 \in B(\mathcal{E}, \mathcal{G})$, with $\|A_0\| < 1$, then F has the radial infimum property if and only if the fractional transform $\Psi_A[F]$ has the radial infimum property.*

Proof. According to [39], the model boundary function \tilde{F} is the limit of $F(rS'_1, \dots, rS'_m)$ in the operator norm, as $r \rightarrow 1$, where S'_1, \dots, S'_m are the left creation operators on the full Fock space $F^2(H_m)$, with m generators. Consequently, for any $\epsilon \in (0, 1)$ there exists $\delta \in (0, 1)$ such that

$$\|F(rS'_1, \dots, rS'_m) - \tilde{F}\| < \epsilon \quad \text{for any } r \in (\delta, 1).$$

Hence, and due to the fact that \tilde{F} is an isometry, we deduce that, for any $r \in (\delta, 1)$ and $x \in F^2(H_m) \otimes \mathcal{E}$,

$$\begin{aligned} \langle F(rS'_1, \dots, rS'_m)^* F(rS'_1, \dots, rS'_m)x, x \rangle^{1/2} &= \|F(rS'_1, \dots, rS'_m)\| \\ &\geq \|\tilde{F}x\| - \|F(rS'_1, \dots, rS'_m) - \tilde{F}\| \\ &\geq \|x\| - \epsilon\|x\| = (1 - \epsilon)\|x\|. \end{aligned}$$

Consequently,

$$F(rS'_1, \dots, rS'_m)^* F(rS'_1, \dots, rS'_m) \geq (1 - \epsilon)^2 I \quad \text{for any } r \in (\delta, 1)$$

and, due to Theorem 3.2, F has the radial infimum property. Therefore, item (i) holds.

To prove (ii), note first that, due to Theorem 1.2, $F \circ \varphi$ is inner. Since $\tilde{F} \in \mathcal{A}_m \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$, and $\tilde{\varphi} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathbb{C}^m, \mathbb{C})$, Theorem 1.1 implies that $\widetilde{F \circ \varphi}$ is in $\mathcal{A}_n \bar{\otimes} B(\mathcal{E}, \mathcal{G})$. Applying now item (i) to $F \circ \varphi$, we deduce part (ii).

Now, we prove (iii). Since $\varphi := [\varphi_1, \dots, \varphi_m]$ is homogeneous of degree $m \geq 1$, we deduce that each φ_j is a homogeneous noncommutative polynomial of degree q . Therefore, $\tilde{\varphi}_j = \varphi_j(S_1, \dots, S_n)$ and

$$(3.4) \quad \varphi_\alpha(rS_1, \dots, rS_n) = r^{q|\alpha|} \varphi_\alpha(S_1, \dots, S_n), \quad \alpha \in \mathbb{F}_m^+,$$

where S_1, \dots, S_n are the left creation operators on the full Fock space $F^2(H_n)$. As in the proof of Theorem 1.2, we have

$$(U \otimes I) \widetilde{F \circ \varphi} = (I_{\mathcal{L}} \otimes \tilde{F})(U \otimes I),$$

where \mathcal{L} is a separable Hilbert space and $U : F^2(H_n) \rightarrow \mathcal{L} \otimes F^2(H_m)$ is a unitary operator. Hence, we deduce that

$$(3.5) \quad (U \otimes I) \widetilde{F_r \circ \varphi} = (I_{\mathcal{L}} \otimes \tilde{F}_r)(U \otimes I), \quad r \in (0, 1),$$

where $F_r(X) := F(rX)$, $X \in [B(\mathcal{H})^m]_{1/r}$. Since F_r is a bounded free holomorphic function on $[B(\mathcal{H})^m]_{1/r}$, we have

$$(3.6) \quad \tilde{F}_r = F_r(S'_1, \dots, S'_m) \quad \text{and} \quad \widetilde{F_r \circ \varphi} = F_r(\varphi_1(S_1, \dots, S_m), \dots, \varphi_m(S_1, \dots, S_n)).$$

Since F has the radial infimum property, for any $\epsilon \in (0, 1)$ there is $\delta \in (0, 1)$ such that

$$F_r(S'_1, \dots, S'_m)^* F_r(S'_1, \dots, S'_m) \geq (1 - \epsilon)I \quad \text{for any } r \in (\delta, 1).$$

Hence, and using relations (3.5) and (3.6), we obtain

$$(3.7) \quad \widetilde{(F_r \circ \varphi)^* F_r \circ \varphi} \geq (1 - \epsilon)I \quad \text{for any } r \in (\delta, 1).$$

On the other hand, due to relation (3.4), we have

$$\begin{aligned}\widetilde{F_r \circ \varphi} &= F(r\varphi_1(S_1, \dots, S_n), \dots, r\varphi_m(S_1, \dots, S_n)) \\ &= F(\varphi_1(r^{1/q}S_1, \dots, r^{1/q}S_n), \dots, \varphi_m(r^{1/q}S_1, \dots, r^{1/q}S_n)) \\ &= (F \circ \varphi)(r'S_1, \dots, r'S_n),\end{aligned}$$

where $r' := r^{1/q}$. Now inequality (3.7) becomes

$$(F \circ \varphi)(r'S_1, \dots, r'S_n)^*(F \circ \varphi)(r'S_1, \dots, r'S_n) \geq (1 - \epsilon)I \quad \text{for any } r' \in (\delta^{1/k}, 1).$$

Applying Theorem 3.2, we conclude that $F \circ \varphi$ has the radial infimum property.

To prove item (iv), assume that F has the radial infimum property. Applying Theorem 1.3 to $\Psi_A[F_r]$, $r \in (0, 1)$, we obtain

$$\begin{aligned}I - \Psi_A[F](rS')^*\Psi_A[F](rS') \\ = D_A[I - F(rS')^*A]^{-1}[I - F(rS')^*F(rS')][I - A^*F(rS')]^{-1}D_A,\end{aligned}$$

where $rS' := (rS'_1, \dots, rS'_m)$. Since F has the radial infimum property, there exist constants $c(r) \in (0, 1]$, $r \in (0, 1)$, with $\lim_{r \rightarrow 1} c(r) = 1$ such that

$$F(rS')^*F(rS') \geq c(r)I.$$

Note also that, since $\|A\| < 1$ and $\|F(rS')\| \leq 1$, we have

$$\begin{aligned}\|[I - F(rS')^*A]^{-1}\| &\leq 1 + \|F(rS')^*A\| + \|F(rS')^*A\|^2 + \dots \\ &\leq 1 + \|A\| + \|A\|^2 + \dots \\ &= \frac{1}{1 - \|A\|}\end{aligned}$$

Using all these relations, we deduce that

$$\Psi_A[F](rS')^*\Psi_A[F](rS') \geq \left[1 - \frac{1 - c(r)}{(1 - \|A\|)^2}\right] I.$$

Since $\lim_{r \rightarrow 1} c(r) = 1$, Theorem 3.2 shows that $\Psi_A[F]$ has the radial infimum property. Now, using the fact that $\Psi_F[\Psi_A[F]] = F$, one can prove the converse. The proof is complete. \square

We remark that under the hypothesis of Theorem 3.5, if $\tilde{F} \in \mathcal{A}_m \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ with $\|\tilde{F}\|_\infty = 1$, then F is inner if and only if it has the radial infimum property. This suggests the following open question. Is there any bounded free holomorphic function with the radial infimum property so that its boundary function is not in the noncommutative disc algebra?

Corollary 3.6. *Any free holomorphic automorphism of the noncommutative ball $[B(\mathcal{H})^n]_1$ has the radial infimum property.*

Proof. According to [45], if $\Psi \in \text{Aut}([B(\mathcal{H})^n]_1)$, the automorphism group of all free holomorphic functions on $[B(\mathcal{H})^n]_1$, then its boundary function $\tilde{\Psi} = [\tilde{\Psi}_1, \dots, \tilde{\Psi}_n]$ is an isometry and $\tilde{\Psi}_i \in \mathcal{A}_n$, the noncommutative disc algebra. Applying Theorem 3.5, part (i), we deduce that Ψ has the radial infimum property. The proof is complete. \square

4. FACTORIZATIONS AND FREE HOLOMORPHIC VERSIONS OF CLASSICAL INEQUALITIES.

In this section we study the class of free holomorphic functions with the radial infimum property in connection with factorizations and noncommutative generalizations of Schwarz's lemma and Harnack's double inequality from complex analysis.

If $A, B \in B(\mathcal{K})$ are selfadjoint operators, we say that $A < B$ if $B - A$ is positive and invertible, i.e., there exists a constant $\gamma > 0$ such that $\langle (B - A)h, h \rangle \geq \gamma \|h\|^2$ for any $h \in \mathcal{K}$. Note that $C \in B(\mathcal{K})$ is a strict contraction ($\|C\| < 1$) if and only if $C^*C < I$.

Theorem 4.1. *Let F, Θ , and G be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E}, \mathcal{G})$, $B(\mathcal{Y}, \mathcal{G})$, and $B(\mathcal{E}, \mathcal{Y})$, respectively, such that*

$$F(X) = \Theta(X)G(X), \quad X \in [B(\mathcal{H})^n]_1.$$

Assume that F is bounded with $\|F\|_\infty \leq 1$ and Θ has the radial infimum property with $\|\Theta\|_\infty = 1$. Then $\|G\|_\infty \leq 1$,

$$F(X)F(X)^* \leq \Theta(X)\Theta(X)^*, \quad X \in [B(\mathcal{H})^n]_1,$$

and

$$\|F(X)\| \leq \|\Theta(X)\|, \quad X \in [B(\mathcal{H})^n]_1.$$

If, in addition, $\|G(0)\| < 1$ and $X_0 \in [B(\mathcal{H})^n]_1$, then:

- (i) $F(X_0)F(X_0)^* < \Theta(X_0)\Theta(X_0)^*$ if and only if $\Theta(X_0)\Theta(X_0)^* > 0$;
- (ii) $\|F(X_0)\| < \|\Theta(X_0)\|$ if and only if $G(X_0) \neq 0$.

Proof. Since Θ has the radial infimum property and $\|\Theta\|_\infty = 1$, Theorem 3.2 shows that there exist constants $c(r) \in (0, 1]$, $r \in (0, 1)$, with $\lim_{r \rightarrow 1} c(r) = 1$ and such that

$$\Theta(rS_1, \dots, rS_n)^* \Theta(rS_1, \dots, rS_n) \geq c(r)I.$$

Consequently, taking into account that $F(rS_1, \dots, rS_n) = \Theta(rS_1, \dots, rS_n)G(rS_1, \dots, rS_n)$ for any $r \in [0, 1)$, we deduce that

$$\|\Theta(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)y\| \geq c(r)\|G(rS_1, \dots, rS_n)y\|$$

for any $y \in F^2(H_n) \otimes \mathcal{E}$. Since $\|\Theta(rS_1, \dots, rS_n)\| \leq \|\Theta\|_\infty = 1$ and $\|F(rS_1, \dots, rS_n)\| \leq 1$, the inequality above implies

$$c(r)\|G(rS_1, \dots, rS_n)\| \leq 1 \quad \text{for any } r \in [0, 1).$$

Using the fact the map $r \mapsto \|G(rS_1, \dots, rS_n)\|$ is increasing and that $\lim_{r \rightarrow 1} c(r) = 1$, we deduce that $\lim_{r \rightarrow 1} \|G(rS_1, \dots, rS_n)\| \leq 1$. Hence, G is bounded and

$$\|G\|_\infty = \lim_{r \rightarrow 1} \|G(rS_1, \dots, rS_n)\| \leq 1.$$

Consequently, $G(X)G(X)^* \leq I$ and

$$F(X)F(X)^* = \Theta(X)G(X)G(X)^*\Theta(X)^* \leq \Theta(X)\Theta(X)^*, \quad X \in [B(\mathcal{H})^n]_1.$$

Hence, we have $F(X)F(X)^* \leq \Theta(X)\Theta(X)^*$ for all $X \in [B(\mathcal{H})^n]_1$.

To prove the second part of this theorem, assume that $\|G(0)\| < 1$. According to Corollary 1.6, we have $\|G(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. Since $F(X) = \Theta(X)G(X)$, $X \in [B(\mathcal{H})^n]_1$, we deduce that

$$(4.1) \quad \Theta(X)\Theta(X)^* - F(X)F(X)^* \geq (1 - \|G(X)\|^2)\Theta(X)\Theta(X)^*.$$

Since $\|G(X)\| < 1$, we have $(1 - \|G(X)\|^2)\Theta(X)\Theta(X)^* \geq 0$. Note also that if $X_0 \in [B(\mathcal{H})^n]_1$ is such that $\Theta(X_0)\Theta(X_0)^* > 0$ then relation (4.1) implies $\Theta(X_0)\Theta(X_0)^* - F(X_0)F(X_0)^* > 0$. The converse is obviously true.

To prove item (ii), note that when $\|G(X_0)\| < 1$ and $G(X_0) \neq 0$, we have

$$\|F(X_0)\| = \|\Theta(X_0)G(X_0)\| \leq \|\Theta(X_0)\|\|G(X_0)\| < \|\Theta(X_0)\|.$$

Consequently, since $F(X_0) = \Theta(X_0)G(X_0)$, we deduce that $\|F(X_0)\| < \|\Theta(X_0)\|$ if and only if $G(X_0) \neq 0$. This completes the proof. \square

Proposition 4.2. *Let F, Θ , and G be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E}, \mathcal{G})$, $B(\mathcal{Y}, \mathcal{G})$, and $B(\mathcal{E}, \mathcal{Y})$, respectively, such that*

$$F(X) = \Theta(X)G(X), \quad X \in [B(\mathcal{H})^n]_1.$$

Assume that:

- (i) $\|F\|_\infty \leq 1$ and $\tilde{F} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$;
- (ii) Θ is inner and $\tilde{\Theta} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y}, \mathcal{G})$.

Then $\|G\|_\infty \leq 1$, $\tilde{G} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{Y})$, and

$$F(X)F(X)^* \leq \Theta(X)\Theta(X)^*, \quad X \in [B(\mathcal{H})^n]_1.$$

If, in addition, F is inner, then so is G .

Proof. Since Θ is inner, i.e., $\tilde{\Theta}^* \tilde{\Theta} = I$, and $\tilde{\Theta} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y}, \mathcal{G})$, Theorem 3.5 implies that Θ has the radial infimum property. Now, due to Theorem 4.1, G is bounded and $\|G\|_\infty \leq 1$. Consequently, inequality $F(X)F(X)^* \leq \Theta(X)\Theta(X)^*$ for all $X \in [B(\mathcal{H})^n]_1$.

On the other hand, since $\tilde{F} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ and $\tilde{\Theta} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{Y}, \mathcal{G})$, according to [39], [44], we have

$$(4.2) \quad \tilde{\Theta} = \lim_{r \rightarrow 1} \Theta(rS_1, \dots, rS_n) \quad \text{and} \quad \tilde{F} = \lim_{r \rightarrow 1} F(rS_1, \dots, rS_n),$$

in the operator norm topology. Since $\|G\|_\infty \leq 1$, its boundary function $\tilde{G} = \text{SOT-}\lim_{r \rightarrow 1} G(rS_1, \dots, rS_n)$ exists. Now, for any $r \in [0, 1)$, we have

$$\Theta(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) = \Theta(rS_1, \dots, rS_n)^* \Theta(rS_1, \dots, rS_n) G(rS_1, \dots, rS_n).$$

Taking the SOT-limit in this equality and using the fact that $\|\Theta(rS_1, \dots, rS_n)\| \leq 1$ and $\|F(rS_1, \dots, rS_n)\| \leq 1$, we deduce that $\tilde{\Theta}^* \tilde{F} = \tilde{G}$. Now, due to relation (4.2), we have

$$\tilde{\Theta}^* \tilde{F} = \lim_{r \rightarrow 1} \Theta(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n) \quad \text{and} \quad \lim_{r \rightarrow 1} \Theta(rS_1, \dots, rS_n)^* \Theta(rS_1, \dots, rS_n) = I$$

in the operator norm. Consequently, since

$$\begin{aligned} \|\tilde{G} - G(rS_1, \dots, rS_n)\| &\leq \|\tilde{G} - \Theta(rS_1, \dots, rS_n)^* F(rS_1, \dots, rS_n)\| \\ &\quad + \|\Theta(rS_1, \dots, rS_n)^* \Theta(rS_1, \dots, rS_n) - I\| \|G(rS_1, \dots, rS_n)\| \end{aligned}$$

and $\|G(rS_1, \dots, rS_n)\| \leq 1$, we deduce that $\tilde{G} = \lim_{r \rightarrow 1} G(rS_1, \dots, rS_n)$ in the operator norm topology. Hence, we deduce that $\tilde{G} \in \mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{Y})$. The proof is complete.

If in addition, F is inner, then relation $\tilde{F} = \tilde{\Theta} \tilde{G}$ implies

$$I = \tilde{F}^* \tilde{F} = \tilde{G}^* \tilde{\Theta}^* \tilde{\Theta} \tilde{G} = \tilde{G}^* \tilde{G},$$

which proves that G is inner. □

We remark that the second part of Theorem 4.1 holds also under the hypothesis of Proposition 4.2.

In [39], [45], and [46], we obtained analogues of Schwarz lemma for free holomorphic functions. We mention the following. Let $F(X) = \sum_{\alpha \in \mathbb{F}_n^+} X_\alpha \otimes A_{(\alpha)}$, $A_{(\alpha)} \in B(\mathcal{E}, \mathcal{G})$, be a free holomorphic function on $[B(\mathcal{H})^n]_1$ with $\|F\|_\infty \leq 1$ and $F(0) = 0$. Then

$$\|F(X)\| \leq \|X\|, \quad \text{for any } X \in [B(\mathcal{H})^n]_1.$$

Note that Theorem 4.1 can be seen as a generalization of Schwarz's lemma. Let us consider a few important particular cases.

Corollary 4.3. *If $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is a bounded free holomorphic function with $\|F\|_\infty \leq 1$ and representation*

$$F(X_1, \dots, X_n) = \sum_{k=m}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)},$$

where $m = 1, 2, \dots$, then

$$F(X_1, \dots, X_n)F(X_1, \dots, X_n)^* \leq \sum_{|\beta|=m} X_\beta X_\beta^* \otimes I_{\mathcal{G}}$$

for any $X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$. If, in addition, $\left\| \sum_{|\beta|=m} A_{(\beta)} A_{(\beta)}^* \right\| < 1$, then the inequality above is strict for any $X \neq 0$.

Proof. As in the the proof of Theorem 3.4 from [39], we have the Gleason type factorization $F = \Theta G$, where Θ and G are free holomorphic functions given by

$$\Theta(X_1, \dots, X_n) := [X_\beta \otimes I_{\mathcal{G}} : |\beta| = m] \quad \text{and} \quad G(X_1, \dots, X_n) = \begin{bmatrix} \Phi_{(\beta)}(X_1, \dots, X_n) \\ \vdots \\ |\beta| = m \end{bmatrix}.$$

Due to Section 3 (see Example 3.3), Θ is inner and has the radial infimum property. Applying now Theorem 4.1, we deduce that

$$F(X_1, \dots, X_n)F(X_1, \dots, X_n)^* \leq \Theta(X_1, \dots, X_n)\Theta(X_1, \dots, X_n)^* = \sum_{|\beta|=m} X_\beta X_\beta^* \otimes I_{\mathcal{G}}$$

for any $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$. On the other hand, since

$$\|G(0)\| = \left\| \begin{bmatrix} A_{(\beta)} \\ \vdots \\ |\beta| = m \end{bmatrix} \right\| = \left\| \sum_{|\beta|=m} A_{(\beta)} A_{(\beta)}^* \right\| < 1,$$

we can use the second part of Theorem 4.1, to complete the proof. \square

We remark that Corollary 4.3 implies the version of Schwarz lemma obtained in [39] and, when $m = 1$, the corresponding result from [20].

Corollary 4.4. *If $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is a bounded free holomorphic function with $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$, then*

$$\begin{aligned} D_{F(0)^*} [I - F(X)F(0)^*]^{-1} [I - F(X)F(X)^*] [I - F(0)F(X)^*]^{-1} D_{F(0)^*} \\ \geq \left(I - \sum_{|\alpha|=1} X_\alpha X_\alpha^* \right) \otimes I_{\mathcal{G}} \end{aligned}$$

for any $X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$.

Proof. According to Theorem 1.3, the mapping $G := \Psi_{F(0)}[F]$ is a bounded free holomorphic function with $\|G\|_\infty \leq 1$ and $G(0) = \Psi_{F(0)}[F(0)] = 0$. Applying Corollary 4.3 to G (when $m = 1$), we deduce that $G(X)G(X)^* \leq XX^* \otimes I_{\mathcal{G}}$. On the other hand, using relation (1.10), we have

$$I - G(X)G(X)^* = D_{F(0)^*} [I - F(X)F(0)^*]^{-1} [I - F(X)F(X)^*] [I - F(0)F(X)^*]^{-1} D_{F(0)^*}.$$

Now, one can easily complete the proof. \square

We remark that Corollary 4.4 can be seen as an extension on Corollary 4.3 (case $m = 1$) to the case when $\|F(0)\| < 1$.

When dealing with free holomorphic functions with scalar coefficients, Theorem 4.1 can be improved, as follows.

Theorem 4.5. *Let f, θ , and g be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with scalar coefficients such that:*

- (i) $f(X) = \theta(X)g(X)$, $X \in [B(\mathcal{H})^n]_1$;
- (ii) f is bounded with $\|f\|_\infty \leq 1$;
- (iii) θ has the radial infimum property and $\|\theta\|_\infty = 1$.

Then $\|g\|_\infty \leq 1$ and, consequently,

$$f(X)f(X)^* \leq \theta(X)\theta(X)^*, \quad X \in [B(\mathcal{H})^n]_1,$$

$$\|f(X)\| \leq \|\theta(X)\|, \quad X \in [B(\mathcal{H})^n]_1,$$

and

$$\|g(0)\| \leq 1.$$

Moreover,

- (a) $f(X_0)f(X_0)^* < \theta(X_0)\theta(X_0)^*$ for some $X_0 \in [B(\mathcal{H})^n]_1$ if and only if $\theta(X_0)\theta(X_0)^* > 0$ and g is not a constant c with $|c| = 1$.
- (b) $\|f(X_0)\| = \|\theta(X_0)\|$ for some $X_0 \in [B(\mathcal{H})^n]_1$ if and only if either $\theta(X_0) = 0$ or $f = c\theta$ for some constant c with $|c| = 1$.
- (c) If $|g(0)| = 1$, then $f = c\theta$ for some constant c with $|c| = 1$.

Proof. Due to Proposition 1.4, if $g : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$ is a non-constant free holomorphic function with $\|g\|_\infty \leq 1$, then $\|g(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. Using this result and Theorem 4.1, in the particular case when $\mathcal{E} = \mathcal{G} = \mathcal{Y} = \mathbb{C}$, one can complete the proof. \square

We remark that in the particular case when $n = 1$ and $\theta(z) = z$, we recover Schwarz's lemma (see [10]).

In what follows we obtain generalizations of some classical results from complex analysis for certain classes of free holomorphic functions with positive real parts and of the form $F = I + \Theta\Gamma$.

Theorem 4.6. *Let F , Θ , and Γ be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$, $B(\mathcal{G}, \mathcal{E})$, and $B(\mathcal{E}, \mathcal{G})$, respectively. If*

- (i) $\Re F \geq 0$,
- (ii) Θ has the radial infimum property, $\|\Theta\|_\infty = 1$, and $\|\Theta(0)\| < 1$,
- (iii) $F = I + \Theta\Gamma$,

then

$$[I - F(X)][I - F(X)^*] \leq [I + F(X)]\Theta(X)\Theta(X)^*[I + F(X)^*]$$

and

$$\|F(X)\| \leq \frac{1 + \|\Theta(X)\|}{1 - \|\Theta(X)\|}$$

for any $X \in [B(\mathcal{H})^n]_1$.

Proof. Since $\Re F(X) \geq 0$, $X \in [B(\mathcal{H})^n]_1$, its noncommutative Cayley transform $G := (F - I)(I + F)^{-1}$ is in the unit ball of $H_{\text{ball}}^\infty(B(\mathcal{E}))$, thus $\|G(X)\| \leq 1$. Due to item (iii), we have $G = \Theta\Gamma(I + F)^{-1}$. Now, since Θ has the radial infimum property and $\|\Theta\|_\infty = 1$, we can apply Theorem 4.1 to G and obtain $G(X)G(X)^* \leq \Theta(X)\Theta(X)^*$ for all $X \in [B(\mathcal{H})^n]_1$. Hence, we deduce that

$$[I + F(X)]^{-1}[F(X) - I][F(X)^* - I][I + F(X)^*]^{-1} \leq \Theta(X)\Theta(X)^*,$$

which is equivalent to

$$[I - F(X)][I - F(X)^*] \leq [I + F(X)]\Theta(X)\Theta(X)^*[I + F(X)^*].$$

The latter inequality implies

$$\|F(X)\| - 1 \leq \|I - F(X)\| \leq \|\Theta(X)\|(1 + \|F(X)\|),$$

which leads to

$$(4.3) \quad \|F(X)\|(1 - \|\Theta(X)\|) \leq 1 + \|\Theta(X)\|.$$

Since $\|\Theta\|_\infty = 1$ and $\|\Theta(0)\| < 1$, the maximum principle for free holomorphic functions with operator-valued coefficients (see Theorem 1.5) implies that $\|\Theta(X)\| < 1$. Now, inequality (4.3) implies the desired inequality. \square

Taking into account Theorem 4.5 and Theorem 4.6, we can make the following observation.

Remark 4.7. *In the scalar case, when $\mathcal{E} = \mathcal{G} = \mathbb{C}$, the first inequality in Theorem 4.6 is strict if and only if $\Theta(X)\Theta(X)^* > 0$ and F is not of the form $F = (I + \eta\Theta)(I - \eta\Theta)^{-1}$ for some constant η with $|\eta| = 1$.*

Consider the set $[H_{\mathbf{ball}}^\infty(B(\mathcal{E}))]_{<1}$ (resp. $H_{\mathbf{ball}}^{>0}(B(\mathcal{E}))$) of all bounded free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ such that $\|F(X)\| < 1$ (resp. $\Re F(X) > 0$) for any $X \in [B(\mathcal{H})^n]_1$. We remark that the restriction to $H_{\mathbf{ball}}^{>0}(B(\mathcal{E}))$ of the noncommutative Cayley transform, defined by

$$\mathcal{C}[F] := [F - 1][1 + F]^{-1},$$

is a bijection $\mathcal{C} : H_{\mathbf{ball}}^{>0}(B(\mathcal{E})) \rightarrow [H_{\mathbf{ball}}^\infty(B(\mathcal{E}))]_{<1}$ and $\mathcal{C}^{-1}[G] = [I + G][I - G]^{-1}$. Indeed, taking into account Theorem 1.5 from [41], it is enough to show that $F \in H_{\mathbf{ball}}^{>0}(B(\mathcal{E}))$ if and only if $G \in [H_{\mathbf{ball}}^\infty(B(\mathcal{E}))]_{<1}$, where $F = [I + G][I - G]^{-1}$. To this end, note that

$$\begin{aligned} 2\Re F(rS_1, \dots, rS_n) \\ = [I - G(rS_1, \dots, rS_n)]^{-1} [I - G(rS_1, \dots, rS_n)^* G(rS_1, \dots, rS_n)] [I - G(rS_1, \dots, rS_n)]^{-1} \end{aligned}$$

for any $r \in [0, 1)$. Consequently, $\Re F(rS_1, \dots, rS_n) > 0$ for any $r \in [0, 1)$ if and only if $\|G(rS_1, \dots, rS_n)\| < 1$ for any $r \in [0, 1)$. Using the noncommutative Poisson transform, we deduce that $\Re F(X) > 0$ for any $X \in [B(\mathcal{H})^n]_1$ if and only if $\|G(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$, which proves our assertion.

In what follows we need the following result.

Lemma 4.8. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ be a free holomorphic function with coefficients in $B(\mathcal{E})$. Then there is a free holomorphic function $G : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ such that*

$$F(X)G(X) = G(X)F(X) = I, \quad X \in [B(\mathcal{H})^n]_1,$$

if and only if $F(rS_1, \dots, rS_n)$ is an invertible operator for any $r \in [0, 1)$. Moreover, in this case,

$$G(rS_1, \dots, rS_n) = F(rS_1, \dots, rS_n)^{-1}, \quad r \in [0, 1),$$

where S_1, \dots, S_n are the left creation operators.

Proof. One implication is obvious. Assume that $F(rS_1, \dots, rS_n)$ is an invertible operator for any $r \in [0, 1)$. First we prove that $F(rS_1, \dots, rS_n)^{-1}$ is in $F_n^\infty \bar{\otimes} B(\mathcal{E})$, the weakly closed algebra generated by the spatial tensor product. Since F is a free holomorphic function, $F(rS_1, \dots, rS_n)$ is in $\mathcal{A}_n \bar{\otimes}_{\min} B(\mathcal{E})$ for any $r \in [0, 1)$. In particular, we have

$$F(rS_1, \dots, rS_n)(R_i \otimes I) = (R_i \otimes I)F(rS_1, \dots, rS_n), \quad i = 1, \dots, n,$$

where R_1, \dots, R_n are the right creation operators. Hence, we deduce that

$$(R_i \otimes I)F(rS_1, \dots, rS_n)^{-1} = F(rS_1, \dots, rS_n)^{-1}(R_i \otimes I) \quad i = 1, \dots, n.$$

According to [36], the commutant of $\{R_i \otimes I, i = 1, \dots, n\}$ is equal to $F_n^\infty \bar{\otimes} B(\mathcal{E})$. Consequently, $F(rS_1, \dots, rS_n)^{-1}$ is in $F_n^\infty \bar{\otimes} B(\mathcal{E})$ and has a unique Fourier representation

$$F(rS_1, \dots, rS_n)^{-1} \sim \sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes r^{|\alpha|} B_{(\alpha)}(r)$$

for some operators $B_{(\alpha)}(r) \in B(\mathcal{E})$. We prove now that the operators $B_{(\alpha)}(r)$, $\alpha \in \mathbb{F}_n^+$, don't depend on $r \in [0, 1)$. Assume that F has the representation

$$F(X_1, \dots, X_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_\alpha, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

where the convergence is in the operator norm topology. Since

$$I = F(rS_1, \dots, rS_n)F(rS_1, \dots, rS_n)^{-1} = \left(\sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes r^{|\alpha|} B_{(\alpha)}(r) \right) \left(\sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} S_\alpha \otimes A_\alpha \right)$$

for any $\beta \in \mathbb{F}_n^+$, we have

$$\langle (S_\beta^* \otimes I_{\mathcal{E}})F(rS_1, \dots, rS_n)F(rS_1, \dots, rS_n)^{-1}(1 \otimes x), (1 \otimes y) \rangle = \sum_{\alpha, \omega \in \mathbb{F}_n^+, \alpha\omega=\beta} r^{|\beta|} \langle A_{(\alpha)}B_{(\omega)}(r)x, y \rangle$$

for any $x, y \in \mathcal{E}$. Therefore, $A_{(0)}B_{(0)}(r) = I$ and

$$(4.4) \quad \sum_{\alpha, \omega \in \mathbb{F}_n^+, \alpha\omega=\beta} A_{(\alpha)}B_{(\omega)}(r) = 0$$

if $|\beta| \geq 1$. Now, we proceed by induction. Note that $B_{(0)}(r) = A_{(0)}^{-1}$ and assume that the operators $B_{(\alpha)}(r)$ don't depend on $r \in [0, 1)$ for any $\alpha \in \mathbb{F}_n^+$ with $|\beta| \leq m$. We prove that the property holds if $|\beta| = m + 1$. To this end, let $\beta := g_{i_1}g_{i_2} \cdots g_{i_m}g_{i_{m+1}} \in \mathbb{F}_n^+$. Due to relation (4.4), we have

$$A_{(0)}B_{(\beta)} + A_{(g_{i_1})}B_{(g_{i_2} \cdots g_{i_{m+1}})} + \cdots + A_{(g_{i_1} \cdots g_{i_m})}B_{(g_{i_{m+1}})} + A_{(\beta)}B_{(0)} = 0.$$

Hence and due to the induction hypothesis, we deduce that $B_{(\beta)}(r)$ does not depend on $r \in [0, 1)$. Thus we can write $B_{(\beta)} := B_{(\beta)}(r)$ for any $\beta \in \mathbb{F}_n^+$ and $F(rS_1, \dots, rS_n)^{-1}$ has the Fourier representation $\sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes r^{|\alpha|} B_{(\alpha)}$ and the series $\sum_{k=0}^{\infty} \sum_{|\alpha|=k} (sr)^{|\alpha|} S_\alpha \otimes B_{(\alpha)}$ converges in the operator norm topology for any $s, r \in [0, 1)$. Hence, we deduce that the map $G : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ defined by

$$G(X_1, \dots, X_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes B_{(\alpha)}, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$$

is a free holomorphic function. Here the convergence is in the operator norm topology. Due to (4.4) and the similar relation that can be deduced from the equation $F(rS_1, \dots, rS_n)^{-1}F(rS_1, \dots, rS_n) = I$, one can easily see that $F(X)G(X) = G(X)F(X) = I$ for any $X \in [B(\mathcal{H})^n]_1$. Moreover, we have

$$G(rS_1, \dots, rS_n) = F(rS_1, \dots, rS_n)^{-1}$$

for any $r \in [0, 1)$. The proof is complete. \square

The next result is a noncommutative extension of Harnack's double inequality.

Theorem 4.9. *Let F , Θ , and Γ be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$, $B(\mathcal{G}, \mathcal{E})$, and $B(\mathcal{E}, \mathcal{G})$, respectively. If*

- (i) $\Re F > 0$,
- (ii) Θ has the radial infimum property, $\|\Theta\|_\infty \leq 1$, and $\|\Theta(0)\| < 1$,
- (iii) $F = I + \Theta\Gamma$ and $F^{-1} = I + \Theta L$ for some free holomorphic function L on $[B(\mathcal{H})^n]_1$,

then

$$\frac{1 - \|\Theta(X)\|}{1 + \|\Theta(X)\|} \leq \|F(X)\| \leq \frac{1 + \|\Theta(X)\|}{1 - \|\Theta(X)\|}$$

for any $X \in [B(\mathcal{H})^n]_1$.

Proof. The inequality $\|F(X)\| \leq \frac{1 + \|\Theta(X)\|}{1 - \|\Theta(X)\|}$ is due to Theorem 4.6. We prove now the first inequality. Since $\Re F(X) > 0$, $X \in [B(\mathcal{H})^n]_1$, there exist constants $\gamma(r) \in (0, 1)$ such that

$$\Re F(rS_1, \dots, rS_n) \geq \gamma(r)I, \quad r \in (0, 1).$$

Hence, we deduce that

$$\|F(rS_1, \dots, rS_n)^*x\| + \|F(rS_1, \dots, rS_n)x\| \geq 2\gamma(r)\|x\|, \quad x \in F^2(H_n) \otimes \mathcal{E},$$

which shows that $F(rS_1, \dots, rS_n)$ and $F(rS_1, \dots, rS_n)^*$ are bounded below. Therefore, the operator $F(rS_1, \dots, rS_n)$ is invertible for all $r \in [0, 1)$. Due to Lemma 4.8, there is a free holomorphic function $\Lambda : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \otimes_{\min} B(\mathcal{E})$ such that

$$F(X)\Lambda(X) = \Lambda(X)F(X) = I, \quad X \in [B(\mathcal{H})^n]_1,$$

and $\Lambda(rS_1, \dots, rS_n) = F(rS_1, \dots, rS_n)^{-1}$ for all $r \in [0, 1)$. Since

$$\Re \Lambda(rS_1, \dots, rS_n) = [F(rS_1, \dots, rS_n)^{-1}]^* [\Re F(rS_1, \dots, rS_n)] F(rS_1, \dots, rS_n)^{-1}$$

and $\Re F(rS_1, \dots, rS_n) > 0$, we deduce that $\Re \Lambda(rS_1, \dots, rS_n) > 0$. Therefore $\Re \Lambda > 0$. Due to item (iii), we have $\Lambda = I + \Theta L$ for some free holomorphic function L on $[B(\mathcal{H})^n]_1$. Applying now Theorem 4.6 to Λ , we obtain

$$\|\Lambda(X)\| \leq \frac{1 + \|\Theta(X)\|}{1 - \|\Theta(X)\|}$$

for any $X \in [B(\mathcal{H})^n]_1$. Since $\Lambda(X) = F(X)^{-1}$, we have $\|F(X)\| \geq \frac{1}{\|\Lambda(X)\|}$. Combining these inequalities, we deduce that

$$\frac{1 - \|\Theta(X)\|}{1 + \|\Theta(X)\|} \leq \|F(X)\|$$

for any $X \in [B(\mathcal{H})^n]_1$. The proof is complete. \square

We remark that when $n = 1$ and $\mathcal{E} = \mathcal{G} = \mathbb{C}$, then the condition $F^{-1} = I + \Theta L$ in Theorem 4.9 is redundant, so we can drop it.

Corollary 4.10. *Let F be a free holomorphic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ and standard representation*

$$F(X_1, \dots, X_n) = I + \sum_{k=m}^{\infty} \sum_{|\alpha|=k} X_{\alpha} \otimes A_{(\alpha)}, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

where $m = 1, 2, \dots$. If $\Re F \geq 0$, then

$$\frac{1 - \|\sum_{|\beta|=m} X_{\beta} X_{\beta}^*\|^{1/2}}{1 + \|\sum_{|\beta|=m} X_{\beta} X_{\beta}^*\|^{1/2}} \leq \|F(X)\| \leq \frac{1 + \|\sum_{|\beta|=m} X_{\beta} X_{\beta}^*\|^{1/2}}{1 - \|\sum_{|\beta|=m} X_{\beta} X_{\beta}^*\|^{1/2}}$$

for any $X \in [B(\mathcal{H})^n]_1$.

Proof. First, we consider the case when $\Re F > 0$. As in the proof of Theorem 2.4 from [39], we have a decomposition $F = I + \Theta \Gamma$, where

$$\Theta(X_1, \dots, X_n) := [X_{\beta} \otimes I : |\beta| = m] \quad \text{and} \quad \Gamma(X_1, \dots, X_n) := \begin{bmatrix} \Phi_{(\beta)}(X_1, \dots, X_n) \\ \vdots \\ |\beta| = m \end{bmatrix}$$

are free holomorphic functions. Due to Section 3 (see Example 3.3), Θ is inner and has the radial infimum property. On the other hand, due to the proof of Theorem 4.9, $X \mapsto F(X)^{-1}$ exists as a free holomorphic function on $[B(\mathcal{H})^n]_1$. Since

$$I = F(X)^{-1}F(X) = F(X)^{-1}(I + \Theta(X)\Gamma(X))$$

we deduce that $F(X)^{-1} = I - F(X)^{-1}\Theta(X)\Gamma(X)$. Taking into account that Θ is a homogeneous polynomial of degree m , it is easy to see that $X \mapsto F(X)^{-1}\Theta(X)\Gamma(X)$ is a free holomorphic function so that each monomial in its standard representation has degree greater than or equal to m . This implies that F^{-1} has a decomposition of the form $I + \Theta L$ for some free holomorphic function L . Since we are under the hypotheses of Theorem 4.9, we can apply this theorem to F and obtain the desired inequalities. In the case when $\Re F \geq 0$, the map $G_{\epsilon} := I + \frac{1}{1+\epsilon}(F - I)$, $\epsilon > 0$, has the property $\Re G_{\epsilon} > 0$, so that we can use the first part of the proof and obtain the corresponding inequalities. Taking the limit as $\epsilon \rightarrow 0$, we complete the proof. \square

From Corollary 4.10, we can deduce the following remarkable particular case, which should be compared to Theorem 1.4 from [47].

Corollary 4.11. *If F is a free holomorphic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ such that $F(0) = I$ and $\Re F \geq 0$, then*

$$\frac{1 - \|X\|}{1 + \|X\|} \leq \|F(X)\| \leq \frac{1 + \|X\|}{1 - \|X\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

5. NONCOMMUTATIVE BOREL-CARATHÉODORY THEOREMS

In this section, we obtain Borel-Carathéodory type results for free holomorphic functions with operator-valued coefficients.

We start with a Carathéodory type result for free holomorphic functions which admit factorizations $F = \Theta\Gamma$, where Θ is an inner function with the radial infimum property.

Theorem 5.1. *Let F , Θ , and Γ be free holomorphic functions on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$, $B(\mathcal{G}, \mathcal{E})$, and $B(\mathcal{E}, \mathcal{G})$, respectively. If*

- (i) $\Re F(X) \leq I$ for any $X \in [B(\mathcal{H})^n]_1$,
- (ii) Θ has the radial infimum property, $\|\Theta\|_\infty = 1$, and $\|\Theta(0)\| < 1$,
- (iii) $F = \Theta\Gamma$,

then

$$\|F(X)\| \leq \frac{2\|\Theta(X)\|}{1 - \|\Theta(X)\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

Proof. Since $G := I - F = I - \Theta\Gamma$ has the property that $\Re G \geq 0$, we can apply Theorem 4.6 to G and obtain

$$[I - G(X)][I - G(X)^*] \leq [I + G(X)]\Theta(X)\Theta(X)^*[I + G(X)^*], \quad X \in [B(\mathcal{H})^n]_1.$$

Consequently, we deduce that

$$\begin{aligned} \|F(X)\| &= \|I - G(X)\| \leq \|(G(X) + I)\|\|\Theta(X)\| \\ &= \|2I - F(X)\|\|\Theta(X)\| \leq (2 + \|F(X)\|)\|\Theta(X)\|. \end{aligned}$$

Hence, we have

$$\|F(X)\|(1 - \|\Theta(X)\|) \leq 2\|\Theta(X)\|.$$

We recall that, since $\|\Theta\|_\infty = 1$ and $\|\Theta(0)\| < 1$, the maximum principle for free holomorphic functions with operator-valued coefficients (see Theorem 1.5) implies that $\|\Theta(X)\| < 1$. Now we can complete the proof. \square

Corollary 5.2. *Let F be a free holomorphic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ and standard representation*

$$F(X_1, \dots, X_n) = \sum_{k=m}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)}, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

where $m = 1, 2, \dots$. If $\Re F \leq I$, then

$$\|F(X)\| \leq \frac{2\|\sum_{|\beta|=m} X_\beta X_\beta^*\|^{1/2}}{1 - \|\sum_{|\beta|=m} X_\beta X_\beta^*\|^{1/2}}, \quad X \in [B(\mathcal{H})^n]_1.$$

Proof. As in the proof of Theorem 2.4 from [39], we have a Gleason type decomposition $F = \Theta\Gamma$, where

$$\Theta(X_1, \dots, X_n) := [X_\beta \otimes I : |\beta| = m] \quad \text{and} \quad \Gamma(X_1, \dots, X_n) := \begin{bmatrix} \Phi_{(\beta)}(X_1, \dots, X_n) \\ \vdots \\ |\beta| = m \end{bmatrix}$$

are free holomorphic functions. Since Θ is inner with the radial infimum property and $\Theta(0) = 0$, we apply Theorem 5.1 and complete the proof. \square

From Corollary 5.2, we can deduce the following particular case.

Corollary 5.3. *If F is a free holomorphic function on $[B(\mathcal{H})^n]_1$ with coefficients in $B(\mathcal{E})$ such that $F(0) = 0$ and $\Re F \leq I$, then*

$$\|F(X)\| \leq \frac{2\|X\|}{1 - \|X\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

The next result is a generalization of the Borel-Carathéodory theorem, mentioned in the introduction, for free holomorphic functions with operator-valued coefficients.

Theorem 5.4. *Let $F : [B(\mathcal{H})^n]_\gamma^- \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ be a free holomorphic function with coefficients in $B(\mathcal{E})$ and let $r \in (0, \gamma)$. Then*

$$\sup_{\|X\|=r} \|F(X)\| \leq \frac{2r}{\gamma - r} A(\gamma) + \frac{\gamma + r}{\gamma - r} \|F(0)\|,$$

where $A(\gamma) := \sup_{\|y\|=1} \langle \Re F(\gamma S_1, \dots, \gamma S_n) y, y \rangle$ and S_1, \dots, S_n are the left creation operators.

Proof. If F is constant, i.e., $F = F(0)$, then the inequality holds due to the fact that

$$\Re F(0) \geq -\|F(0)\| I_{\mathcal{H} \otimes \mathcal{E}}.$$

Assume that F is not constant and $F(0) = 0$. First we show that

$$(5.1) \quad A(\gamma) > 0.$$

Indeed, if we assume that $A(\gamma) \leq 0$, then

$$\Re F(\gamma S_1, \dots, \gamma S_n) \leq \Re F(0) = 0.$$

Applying the noncommutative Poisson transform at $[\frac{t}{\gamma} X_1, \dots, \frac{t}{\gamma} X_n]$, where $0 \leq t < \gamma$ and $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1^-$, we obtain

$$\Re F(tX_1, \dots, tX_n) = P_{[\frac{t}{\gamma} X_1, \dots, \frac{t}{\gamma} X_n]} \Re F(\gamma S_1, \dots, \gamma S_n) \leq \Re F(0) = 0$$

for any $t \in [0, \gamma)$ and $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1^-$. According to Theorem 2.9 from [44], we deduce that $F = F(0)$, which contradicts our assumption. Therefore, inequality (5.1) holds.

Since $\Re F(\gamma S_1, \dots, \gamma S_n) \leq A(\gamma)I$, we can use again the noncommutative Poisson transform to deduce that $\Re F(X) \leq A(\gamma)I$ for $X \in [B(\mathcal{H})^n]_\gamma^-$. Now, let $\epsilon > 0$ and define the free holomorphic function on a noncommutative ball $[B(\mathcal{H})^n]_s$ with $s > \gamma$, by

$$\varphi_\epsilon(X) := 2[A(\gamma) + \epsilon] I_{\mathcal{H} \otimes \mathcal{E}} - F(X), \quad X \in [B(\mathcal{H})^n]_s.$$

Note that, for any $y \in \mathcal{H} \otimes \mathcal{E}$ and $X \in [B(\mathcal{H})^n]_s$, we have

$$\begin{aligned} \|\varphi_\epsilon(X)y\|^2 &= 4[A(\gamma) + \epsilon]^2 \|y\|^2 - 4[A(\gamma) + \epsilon] \langle \Re F(X)y, y \rangle + \|F(X)y\|^2 \\ (5.2) \quad &\geq 4\epsilon[A(\gamma) + \epsilon] \|y\|^2 + \|F(X)y\|^2 \\ &\geq 4\epsilon[A(\gamma) + \epsilon] \|y\|^2. \end{aligned}$$

Similar calculations show that

$$\|\varphi_\epsilon(X)^*y\|^2 \geq 4\epsilon[A(\gamma) + \epsilon] \|y\|^2.$$

Replacing X by (tS_1, \dots, tS_n) , $t \leq s$, and taking $y \in F^2(H_n) \otimes \mathcal{E}$ in the inequalities above, we deduce that $\varphi_\epsilon(tS_1, \dots, tS_n)$ and $\varphi_\epsilon(tS_1, \dots, tS_n)^*$ are bounded below and, consequently, invertible for any $t \in [0, s)$.

Applying Lemma 4.8 to φ_ϵ , we deduce that there is a free holomorphic function ψ_ϵ on $[B(\mathcal{H})^n]_s$ such that

$$\varphi_\epsilon(X)\psi_\epsilon(X) = \psi_\epsilon(X)\varphi_\epsilon(X) = I, \quad X \in [B(\mathcal{H})^n]_s.$$

Using relation (5.2) and replacing y with $\psi_\epsilon(X)y$, we obtain that

$$\|y\|^2 = \|\varphi_\epsilon(X)\psi_\epsilon(X)y\| \geq \|F(X)\psi_\epsilon(X)y\|^2 + 4\epsilon[A(\gamma) + \epsilon] \|\psi_\epsilon(X)y\|^2.$$

Hence, we deduce that the map

$$(5.3) \quad \Lambda_\epsilon(X) := F(X)\psi_\epsilon(X), \quad X \in [B(\mathcal{H})^n]_s,$$

is a contractive free holomorphic function on $[B(\mathcal{H})^n]_s$. Since $\Lambda_\epsilon(0) = 0$, Theorem 1.5 implies that $\|\Lambda_\epsilon(X)\| < 1$. Hence, and due to relation (5.3), we deduce that

$$(5.4) \quad F(X) = 2[A(\gamma) + \epsilon]\Lambda_\epsilon[I + \Lambda_\epsilon(X)]^{-1},$$

which implies

$$\begin{aligned} \|F(X)\| &\leq 2[A(\gamma) + \epsilon]\|\Lambda_\epsilon(X)\| (1 + \|\Lambda_\epsilon(X)\| + \|\Lambda_\epsilon(X)\|^2 + \dots) \\ &\leq 2[A(\gamma) + \epsilon] \frac{\|\Lambda_\epsilon(X)\|}{1 - \|\Lambda_\epsilon(X)\|}. \end{aligned}$$

On the other hand, applying the Schwarz type lemma for free holomorphic functions (see [39]) to Λ_ϵ , we deduce that

$$(5.5) \quad \|\Lambda_\epsilon(X)\| \leq \frac{r}{\gamma}$$

for any $X \in [B(\mathcal{H})^n]_\gamma$ with $\|X\| = r$, where $0 \leq r < \gamma$. Combining this with the previous inequality, we obtain

$$\|F(X)\| \leq \frac{2[A(\gamma) + \epsilon]r}{\gamma - r}.$$

Taking $\epsilon \rightarrow 0$, we deduce that

$$\sup_{\|X\|=r} \|F(X)\| \leq \frac{2r}{\gamma - r} A(\gamma),$$

which proves the theorem when $F(0) = 0$.

Now, we consider the case when $F(0) \neq 0$. Applying the result above to $F - F(0)$, we obtain

$$\begin{aligned} \sup_{\|X\|=r} \|F(X) - F(0)\| &\leq \frac{2r}{\gamma - r} \sup_{\|y\|=1} \langle \Re(F(\gamma S_1, \dots, \gamma S_n) - F(0))y, y \rangle \\ &\leq \frac{2r}{\gamma - r} [A(\gamma) + \|F(0)\|]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \sup_{\|X\|=r} \|F(X)\| &\leq \sup_{\|X\|=r} \|F(X) - F(0)\| + \|F(0)\| \\ &= \frac{2r}{\gamma - r} A(\gamma) + \frac{\gamma + r}{\gamma - r} \|F(0)\|. \end{aligned}$$

The proof is complete. \square

We remark that if $A(\gamma) \geq 0$ in Theorem 5.4, then we can deduce that

$$\sup_{\|X\|=r} \|F(X)\| \leq \frac{\gamma + r}{\gamma - r} [A(\gamma) + \|F(0)\|].$$

A closer look at the proof of Theorem 5.4 reveals another Carathéodory type inequality. More precisely, applying Corollary 4.3 to the free holomorphic function Λ_ϵ , we deduce that

$$\Lambda_\epsilon(X)\Lambda_\epsilon(X)^* \leq \frac{XX^*}{\gamma^2}$$

for any $X \in [B(\mathcal{H})^n]_\gamma$ with $\|X\| = r$, where $0 \leq r < \gamma$. Using now relations (5.4) and (5.5), we obtain

$$\begin{aligned} F(X)F(X)^* &\leq 4[A(\gamma) + \epsilon]^2 \frac{\Lambda_\epsilon(X)\Lambda_\epsilon(X)^*}{(1 - \|\Lambda_\epsilon(X)\|)^2} \\ &\leq \frac{4[A(\gamma) + \epsilon]^2}{(\gamma - r)^2} (XX^* \otimes I_{\mathcal{E}}). \end{aligned}$$

Taking $\epsilon \rightarrow 0$, we deduce that $F(X)F(X)^* \leq \frac{4A(\gamma)^2}{(\gamma - r)^2} (XX^* \otimes I_{\mathcal{E}})$. Now, in the general case when $F(0)$ is not necessarily 0, we obtain the following result

Corollary 5.5. *Under the hypotheses of Theorem 5.4, we have*

$$[F(X) - F(0)][F(X) - F(0)]^* \leq \frac{4A(\gamma)^2}{(\gamma - r)^2} (XX^* \otimes I_{\mathcal{E}})$$

for any $X \in B(\mathcal{H})^n$ with $\|X\| = r$.

6. JULIA'S LEMMA FOR HOLOMORPHIC FUNCTIONS ON NONCOMMUTATIVE BALLS

In this section, we provide a noncommutative generalization of Pick's theorem for bounded free holomorphic functions. Using this result and basic facts concerning the involutive free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$, we obtain a free analogue of Julia's lemma from complex analysis.

A map $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^n]_1$ is called free biholomorphic if F is free holomorphic, one-to-one and onto, and has free holomorphic inverse. The automorphism group of $[B(\mathcal{H})^n]_1$, denoted by $\text{Aut}([B(\mathcal{H})^n]_1)$, consists of all free biholomorphic functions of $[B(\mathcal{H})^n]_1$. It is clear that $\text{Aut}([B(\mathcal{H})^n]_1)$ is a group with respect to the composition of free holomorphic functions.

Inspired by the classical results of Siegel [52] and Phillips [30] (see also [57]), we used, in [45], the theory of noncommutative characteristic functions for row contractions (see [33]) to find all the involutive free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$, which turn out to be of the form

$$\Phi_\lambda(X_1, \dots, X_n) = -\Theta_\lambda(X_1, \dots, X_n), \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

for some $\lambda = [\lambda_1, \dots, \lambda_n] \in \mathbb{B}_n$, where Θ_λ is the characteristic function of the row contraction λ , acting as an operator from \mathbb{C}^n to \mathbb{C} .

We recall that the characteristic function of the row contraction $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$ is the boundary function $\tilde{\Theta}_\lambda$, with respect to R_1, \dots, R_n , of the free holomorphic function $\Theta_\lambda : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^n]_1$ given by

$$\Theta_\lambda(X_1, \dots, X_n) := -\lambda + \Delta_\lambda \left(I_{\mathcal{H}} - \sum_{i=1}^n \bar{\lambda}_i X_i \right)^{-1} [X_1, \dots, X_n] \Delta_\lambda^*$$

for $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$, where $\Delta_\lambda = (1 - \|\lambda\|_2^2)^{1/2} I_{\mathbb{C}}$ and $\Delta_\lambda^* = (I_{\mathcal{K}} - \lambda^* \lambda)^{1/2}$. For simplicity, we also used the notation $\lambda := [\lambda_1 I_{\mathcal{G}}, \dots, \lambda_n I_{\mathcal{G}}]$ for the row contraction acting from $\mathcal{G}^{(n)}$ to \mathcal{G} , where \mathcal{G} is a Hilbert space.

In [45], we proved that if $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n \setminus \{0\}$ and $\gamma := \frac{1}{\|\lambda\|_2}$, then $\Phi_\lambda := -\Theta_\lambda$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ which has the following properties:

- (i) $\Phi_\lambda(0) = \lambda$ and $\Phi_\lambda(\lambda) = 0$;
- (ii) the identities

$$(6.1) \quad \begin{aligned} I_{\mathcal{H}} - \Phi_\lambda(X) \Phi_\lambda(Y)^* &= \Delta_\lambda (I - X \lambda^*)^{-1} (I - X Y^*) (I - \lambda Y^*)^{-1} \Delta_\lambda, \\ I_{\mathcal{H} \otimes \mathbb{C}^n} - \Phi_\lambda(X)^* \Phi_\lambda(Y) &= \Delta_\lambda^* (I - X^* \lambda)^{-1} (I - X^* Y) (I - \lambda^* Y)^{-1} \Delta_\lambda^*, \end{aligned}$$

hold for all X and Y in $[B(\mathcal{H})^n]_\gamma$;

- (iii) Φ_λ is an involution, i.e., $\Phi_\lambda(\Phi_\lambda(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$;
- (iv) Φ_λ is a free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^n]_1$;
- (v) Φ_λ is a homeomorphism of $[B(\mathcal{H})^n]_1^-$ onto $[B(\mathcal{H})^n]_1^-$.

Moreover, we determined all the free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$ by showing that if $\Phi \in \text{Aut}([B(\mathcal{H})^n]_1)$ and $\lambda := \Phi^{-1}(0)$, then there is a unitary operator U on \mathbb{C}^n such that

$$\Phi = \Phi_U \circ \Phi_\lambda,$$

where

$$\Phi_U(X_1, \dots, X_n) := [X_1, \dots, X_n] U, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1.$$

The first result of this section is following extension of Pick's theorem (see [31], [9]), for bounded free holomorphic functions. Let $M_{n \times m}$ be the set of all $n \times m$ matrices with scalar coefficients.

Theorem 6.1. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ be a free holomorphic function with $\|F(0)\| < 1$ and let $a \in \mathbb{B}_n$. Then there exists a free holomorphic function $\Gamma : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \otimes_{\min} M_{n \times m}$ with $\|\Gamma\|_\infty \leq 1$ such that*

$$\Phi_{F(a)}(F(X)) = \Phi_a(X)(\Gamma \circ \Phi_a)(X), \quad X \in [B(\mathcal{H})^n]_1,$$

where Φ_a and $\Phi_{F(a)}$ are the corresponding free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$ and $[B(\mathcal{H})^m]_1$, respectively. Consequently,

$$\Phi_{F(a)}(F(X))\Phi_{F(a)}(F(X))^* \leq \Phi_a(X)\Phi_a(X)^*, \quad X \in [B(\mathcal{H})^n]_1,$$

and

$$\|\Phi_{F(a)}(F(X))\| \leq \|\Phi_a(X)\|, \quad X \in [B(\mathcal{H})^n]_1.$$

Proof. Since F is a free holomorphic function with $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$, Corollary 1.6 implies that $\|F(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. We know that $\Phi_a \in \text{Aut}([B(\mathcal{H})^n]_1)$ and $\Phi_{F(a)} \in \text{Aut}([B(\mathcal{H})^m]_1)$. Due to Section 1 and the properties of the free holomorphic automorphisms of $[B(\mathcal{H})^m]_1$, the composition map $G := \Phi_{F(a)} \circ F \circ \Phi_a : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1$ is a free holomorphic function with $G(0) = 0$. Therefore, it has a representation of the form

$$(6.2) \quad G(X_1, \dots, X_n) = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)} = [X_1, \dots, X_n] \Gamma(X_1, \dots, X_n)$$

for any $[X_1, \dots, X_n] \in [B(\mathcal{H})^n]_1$, for some matrices $A_{(\alpha)} \in M_{1 \times m}$ and a free holomorphic function Γ with coefficients in $M_{n \times m}$. Since $\|G\|_\infty \leq 1$ with $G(0) < 1$, and $\Theta(X) := [X_1, \dots, X_n]$ is inner and has the radial infimum property, Theorem 4.1 implies that $\|\Gamma\|_\infty \leq 1$,

$$G(X)G(X)^* \leq XX^*, \quad X \in [B(\mathcal{H})^n]_1,$$

and

$$\|G(X)\| \leq \|X\|, \quad X \in [B(\mathcal{H})^n]_1.$$

Replacing X by $\Phi_a(X)$ in these inequalities and in relation (6.2), and using the fact that $\Phi_a \circ \Phi_a = \text{id}$, we complete the proof. \square

Corollary 6.2. *If $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ is a free holomorphic function with $\|F(0)\| < 1$ and $a \in \mathbb{B}_n$, then, for any $X \in [B(\mathcal{H})^n]_1$,*

$$(6.3) \quad \begin{aligned} & \Delta_{F(a)}[I - F(X)F(a)^*]^{-1}[I - F(X)F(X)^*][I - F(a)F(X)^*]^{-1}\Delta_{F(a)} \\ & \geq \Delta_a(I - Xa^*)^{-1}(I - XX^*)(I - aX^*)^{-1}\Delta_a \end{aligned}$$

and

$$(6.4) \quad \begin{aligned} & \|[I - F(a)F(X)^*][I - F(X)F(X)^*]^{-1}[I - F(X)F(a)^*]\| \\ & \leq \frac{\Delta_{F(a)}^2}{\Delta_a^2} \|(I - a^*X)(I - XX^*)^{-1}(I - Xa^*)\|. \end{aligned}$$

Proof. The first inequality follows from Theorem 6.1 and relation (6.1). Since $\|F(0)\| < 1$, Corollary 1.6 implies that $\|F(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. Note that each side of inequality (6.3) is a positive invertible operator. It is well-known that if A, B are two positive invertible operator such that $A \leq B$ then $B^{-1} \leq A^{-1}$. Applying this result to inequality (6.3), we deduce that

$$\begin{aligned} & [I - F(a)F(X)^*][I - F(X)F(X)^*]^{-1}[I - F(X)F(a)^*] \\ & \leq \frac{\Delta_{F(a)}^2}{\Delta_a^2} (I - aX^*)(I - XX^*)^{-1}(I - Xa^*). \end{aligned}$$

Hence, the second inequality follows. The proof is complete. \square

Let $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ be a free holomorphic function. Let $\xi \in \partial \mathbb{B}_n := \{z \in \mathbb{C}^n : \|z\|_2 = 1\}$ and assume that

$$L := \liminf_{z \rightarrow \xi} \frac{1 - \|F(z)\|^2}{1 - \|z\|^2} < \infty.$$

Then there is a sequence $\{z_k\}_{k=1}^\infty \subset \mathbb{B}_n$ such that $\lim_{k \rightarrow \infty} z_k = \xi$ and $\lim_{k \rightarrow \infty} F(z_k) = \eta$ for some $\eta \in \partial \mathbb{B}_m$, and

$$\lim_{k \rightarrow \infty} \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} = L.$$

Now, we can present our first generalization of Julia's lemma for free holomorphic functions on non-commutative balls.

Theorem 6.3. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ be a free holomorphic function with $\|F(0)\| < 1$. Let $\{z_k\}_{k=1}^\infty \subset \mathbb{B}_n$ be a sequence such that $\lim_{k \rightarrow \infty} z_k = \xi$, $\lim_{k \rightarrow \infty} F(z_k) = \eta$ for some $\xi \in \partial \mathbb{B}_n$, $\eta \in \partial \mathbb{B}_m$, and*

$$\lim_{k \rightarrow \infty} \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} = L < \infty.$$

Then the following statements hold.

(i) *For any $X \in [B(\mathcal{H})^n]_1$,*

$$\begin{aligned} & \| [I - \eta F(X)^*] [I - F(X)F(X)^*]^{-1} [I - F(X)\eta^*] \| \\ & \leq L \| (I - \xi^* X)(I - XX^*)^{-1} (I - X\xi^*) \|. \end{aligned}$$

(ii) *If $\beta > 0$ and $X \in [B(\mathcal{H})^n]_1$ is such that*

$$(I - X\xi^*)(I - \xi X^*) < \beta(I - XX^*),$$

then

$$[I - F(X)\eta^*][I - \eta F(X)^*] < \beta L [I - F(X)F(X)^*].$$

Proof. Due to inequality (6.4), we have

$$\begin{aligned} & \| [I - F(z_k)F(X)^*] [I - F(X)F(X)^*]^{-1} [I - F(X)F(z_k)^*] \| \\ & \leq \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} \| (I - z_k^* X)(I - XX^*)^{-1} (I - Xz_k^*) \|. \end{aligned}$$

Taking the limit as $k \rightarrow \infty$, we deduce item (i). To prove (ii), note first that, using the same inequality (6.4), when $a = z_k$ and $X = 0$, we obtain

$$\begin{aligned} & \| [I - F(z_k)F(0)^*] [I - F(0)F(0)^*]^{-1} [I - F(0)F(z_k)^*] \| \\ & \leq \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2}. \end{aligned}$$

Taking $z_k \rightarrow \xi$ and due to the fact that $\|F(0)\| < 1$, we deduce that

$$L \geq \| [I - \eta F(0)^*][I - F(0)F(0)^*]^{-1} [I - F(0)\eta^*] \| > 0.$$

Notice also that, for any $X \in [B(\mathcal{H})^n]_1$, the following inequalities are equivalent:

- (a) $(I - X\xi^*)(I - \xi X^*) < \beta(I - XX^*),$
- (b) $\| (I - \xi X^*)(I - XX^*)^{-1} (I - X\xi^*) \| < \beta.$

Indeed, inequality (b) holds if and only if $\| (I - \xi X^*)(I - XX^*)^{-1/2} \| < \beta^{1/2}$, which is equivalent to

$$(I - XX^*)^{-1/2} (I - X\xi^*)(I - \xi X^*)(I - XX^*)^{-1/2} < \beta.$$

The latter inequality is clearly equivalent to (a).

Now, to prove (ii), we assume that

$$(I - X\xi^*)(I - \xi X^*) < \beta(I - XX^*).$$

Due to the equivalence of (a) with (b), and using the inequality from (i) and the fact that $L > 0$, we obtain

$$\|[I - \eta F(X)^*][I - F(X)F(X)^*]^{-1}[I - F(X)\eta^*]\| < \beta L.$$

Once again using the equivalence of (a) with (b) when X is replaced by $F(X)$, we obtain that

$$[I - F(X)\eta^*][I - \eta F(X)^*] < \beta L[I - F(X)F(X)^*].$$

This completes the proof. \square

We mention that, using unitary transformations in $B(\mathbb{C}^n)$ and $B(\mathbb{C}^m)$, respectively, we can choose the coordinates such that $\xi = (1, 0, \dots, 0) \in \partial \mathbb{B}_n$ and $\eta = (1, 0, \dots, 0) \in \partial \mathbb{B}_m$, in Theorem 6.3. For $0 < c < 1$, we define the noncommutative ellipsoid

$$\mathbf{E}_c := \left\{ (X_1, \dots, X_n) \in B(\mathcal{H})^n : \frac{[X_1 - (1-c)I][X_1^* - (1-c)I]}{c^2} + \frac{X_2 X_2}{c} + \dots + \frac{X_n X_n}{c} < I \right\}$$

with center at $((1-c)I, 0, \dots, 0)$.

Here is our second version of Julia's lemma for free holomorphic functions.

Theorem 6.4. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1$ be a free holomorphic function. Let $z_k \in \mathbb{B}_n$ be such that $\lim_{k \rightarrow \infty} z_k = (1, 0, \dots, 0) \in \partial \mathbb{B}_n$, $\lim_{k \rightarrow \infty} F(z_k) = (1, 0, \dots, 0) \in \partial \mathbb{B}_m$, and*

$$\lim_{k \rightarrow \infty} \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} = L < \infty.$$

If $F := (F_1, \dots, F_m)$, then the following statements hold.

(i) *For any $X := (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1$,*

$$(I - F_1(X)^*)(I - F(X)F(X)^*)^{-1}(I - F_1(X)) \leq L(I - X_1^*)(I - XX^*)^{-1}(I - X_1).$$

(ii) *If $0 < c < 1$, then*

$$F(\mathbf{E}_c) \subset \mathbf{E}_\gamma, \quad \text{where } \gamma := \frac{Lc}{1 + Lc - c}.$$

Proof. As in the proof of Corollary 6.2, inequality (6.3) implies

$$\begin{aligned} & [I - F(z_k)F(X)^*][I - F(X)F(X)^*]^{-1}[I - F(X)F(z_k)^*] \\ & \leq \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} (I - z_k X^*)(I - XX^*)^{-1}(I - X z_k^*). \end{aligned}$$

Taking the limit as $k \rightarrow \infty$, we obtain the inequality in item (i). Now we prove item (ii). Straightforward calculations reveal that $X = (X_1, \dots, X_n)$ is in the noncommutative ellipsoid \mathbf{E}_c if and only if

$$(6.5) \quad (I - X_1)(I - X_1^*) < \frac{c}{1-c}(I - XX^*).$$

According to the equivalence (a) \leftrightarrow (b) (see the proof of Theorem 6.3), when $\xi = (1, 0, \dots, 0)$ and $\beta := \frac{c}{1-c}$, the latter inequality is equivalent to

$$\|(I - X_1^*)(I - XX^*)^{-1}(I - X_1)\| < \frac{c}{1-c},$$

which is equivalent to

$$(I - X_1^*)(I - XX^*)^{-1}(I - X_1) < \frac{c}{1-c}I.$$

Usind the inequality from item (i), we obtain

$$(I - F_1(X)^*)(I - F(X)F(X)^*)^{-1}(I - F_1(X)) < \frac{Lc}{1-c}I = \frac{\gamma}{1-\gamma}I,$$

where $\gamma := \frac{Lc}{1+Lc-c}$. As above, the latter inequality is equivalent to

$$(I - F_1(X))(I - F_1(X)^*) < \frac{\gamma}{1-\gamma}(I - F(X)F(X)^*),$$

which is equivalent to $F(X) \in \mathbf{E}_\gamma$. This completes the proof. \square

We introduce noncommutative Korany type regions in $[B(\mathcal{H})^n]_1$. For each $\xi \in \partial \mathbb{B}_n$ and $\alpha > 1$, we define

$$\mathbf{D}_\alpha(\xi) := \left\{ X \in B(\mathcal{H})^n : (I - X\xi^*)(I - \xi X^*) < \frac{\alpha^2}{4}(1 - \|X\|^2)(I - XX^*) \right\}.$$

Note that if $\mathcal{H} = \mathbb{C}$, then $\mathbf{D}_\alpha(\xi)$ coincides with the Korany region (see [51])

$$D_\alpha(\xi) = \left\{ z \in \mathbb{C}^n : |1 - \langle z, \xi \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

Corollary 6.5. *If F is as in Theorem 6.4 and $F(0) = 0$, then*

- (i) $F(\mathbf{E}_c) \subset \mathbf{E}_{cL}$, for $0 < c < \frac{1}{L}$;
- (ii) $F(\mathbf{D}_\alpha) \subset \mathbf{D}_{\alpha\sqrt{L}}$, for $\alpha > 1$, where $\mathbf{D}_\alpha = \mathbf{D}_\alpha(1, 0, \dots, 0)$.

Proof. Since $F(0) = 0$, due to Schwarz lemma for free holomorphic functions, we have $\|F(X)\| \leq \|X\|$ for all $X \in [B(\mathcal{H})^n]_1$. Consequently,

$$L = \lim_{k \rightarrow \infty} \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} \geq 1$$

which implies $\gamma := \frac{Lc}{1+Lc-c} \leq Lc$, therefore $\mathbf{E}_\gamma \subset \mathbf{E}_{cL}$. Due to Theorem 6.4, we deduce that $F(\mathbf{E}_c) \subseteq \mathbf{E}_{cL}$ when $0 < c < \frac{1}{L}$.

To prove item (ii), let $X = (X_1, \dots, X_n) \in \mathbf{D}_\alpha$, i.e.,

$$(I - X_1)(I - X_1^*) < \frac{\alpha^2}{4}(1 - \|X\|^2)(I - XX^*).$$

Applying Theorem 6.3, part (ii), when $\xi = (1, 0, \dots, 0)$ and $\beta = \frac{\alpha^2}{4}(1 - \|X\|^2)$ we deduce that

$$[I - F_1(X)][I - F_1(X)^*] < \frac{L\alpha^2}{4}(1 - \|X\|^2)[I - F(X)F(X)^*].$$

Since $\|F(X)\| \leq \|X\|$, we obtain

$$[I - F_1(X)][I - F_1(X)^*] < \frac{L\alpha^2}{4}(1 - \|F(X)\|^2)[I - F(X)F(X)^*],$$

which shows that $F(X) \in \mathbf{D}_{\alpha\sqrt{L}}$ and completes the proof. \square

7. PICK-JULIA THEOREMS FOR FREE HOLOMORPHIC FUNCTIONS WITH OPERATOR-VALUED COEFFICIENTS

In this section, we use fractional transforms and a version of the noncommutative Schwarz's lemma to obtain Pick-Julia theorems for free holomorphic functions F with operator-valued coefficients such that $\|F\|_\infty \leq 1$ (resp. $\Re F \geq 0$). As a consequence, we obtain a Julia type lemma for free holomorphic functions with positive real parts. We also provide commutative versions of these results for operator-valued multipliers of the Drury-Arveson space.

Theorem 7.1. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ be a free holomorphic function with $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$. If $z \in \mathbb{B}_n$, then*

$$\Psi_{F(z)}(F(X))\Psi_{F(z)}(F(X))^* \leq \Phi_z(X)\Phi_z(X)^* \otimes I_{\mathcal{G}}, \quad X \in [B(\mathcal{H})^n]_1,$$

where $\Psi_{F(z)}$ is the fractional transform defined by (1.6) and Φ_z is the corresponding free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$. Moreover, we have

$$\begin{aligned} D_{F(z)^*}[I - F(X)F(z)^*]^{-1}[I - F(X)F(X)^*][I - F(z)F(X)^*]^{-1}D_{F(z)^*} \\ \geq \Delta_z(I - Xz^*)^{-1}(I - XX^*)(I - zX^*)^{-1}\Delta_z \otimes I_{\mathcal{G}} \end{aligned}$$

for any $z \in \mathbb{B}_n$ and $X \in [B(\mathcal{H})^n]_1$.

Proof. Since $\|F(0)\| < 1$, Corollary 1.6 implies that $\|F(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. According to Theorem 1.1 and Theorem 1.3, the mapping $X \mapsto (\Psi_{F(z)} \circ F \circ \Phi_z)(X)$ is a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$ with $\|(\Psi_{F(z)} \circ F \circ \Phi_z)(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. On the other hand, since we have $(\Psi_{F(z)} \circ F \circ \Phi_z)(0) = \Psi_{F(z)}(F(z)) = 0$, we can apply Corollary 4.3 and obtain

$$(\Psi_{F(z)} \circ F \circ \Phi_z)(Y)[(\Psi_{F(z)} \circ F \circ \Phi_z)(Y)]^* \leq YY^* \otimes I_G$$

for any $Y \in [B(\mathcal{H})^n]_1$. Taking $Y = \Phi_z(X)$, $X \in [B(\mathcal{H})^n]_1$, and due to the identity $\Phi_z \circ \Phi_z = \text{id}$, we obtain

$$\Psi_{F(z)}(F(X))\Psi_{F(z)}(F(X))^* \leq \Phi_z(X)\Phi_z(X)^* \otimes I_G, \quad X \in [B(\mathcal{H})^n]_1.$$

Using now relations (1.7) and (6.1), we complete the proof. \square

We remark that under the conditions of Theorem 7.1, one can show, as in the proof of Theorem 6.1, that there is a free holomorphic function G with operator-valued coefficients and $\|G\|_\infty \leq 1$ such that

$$\Psi_{F(z)}[F(X)] = \Phi_z(X)(G \circ \Phi_z)(X), \quad X \in [B(\mathcal{H})^n]_1.$$

Our Pick-Julia type result for free holomorphic functions with positive real parts is the following.

Theorem 7.2. *If $G : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ is a free holomorphic function with $\Re G > 0$, then*

$$\begin{aligned} \Gamma(z)[I + G(z)^*][G(X) + G(z)^*]^{-1}[\Re G(X)][G(z) + G(X)^*]^{-1}[I + G(z)]\Gamma(z) \\ \geq (1 - \|z\|_2^2)(I - Xz^*)^{-1}(I - XX^*)(I - zX^*)^{-1} \otimes I_{\mathcal{E}} \end{aligned}$$

for any $z \in \mathbb{B}_n$ and $X \in [B(\mathcal{H})^n]_1$, where

$$\Gamma(z) := 2 \{ [I + G(z)]^{-1}[\Re G(z)][I + G(z)^*]^{-1} \}^{1/2}.$$

Proof. According to the considerations preceding Lemma 4.8, since $\Re G > 0$, the noncommutative Cayley transform $F := \mathcal{C}[G] := (G - I)(I + G)^{-1}$ is a bounded free holomorphic function with $\|F(X)\| < 1$ for any $X \in [B(\mathcal{H})^n]_1$. Due to Theorem 7.1, we obtain

$$\begin{aligned} (7.1) \quad D_{F(z)^*}[I - F(X)F(z)^*]^{-1}[I - F(X)F(X)^*][I - F(z)F(X)^*]^{-1}D_{F(z)^*} \\ \geq \Delta_z(I - Xz^*)^{-1}(I - XX^*)(I - zX^*)^{-1}\Delta_z \otimes I_G. \end{aligned}$$

Note that

$$\begin{aligned} I - F(X)F(z)^* &= I - [I + G(X)]^{-1}[G(X) - I][G(z)^* - I][I + G(z)^*]^{-1} \\ &= [I + G(X)]^{-1} \{ [I + G(X)][I + G(z)^*] - [G(X) - I][G(z)^* - I] \} [I + G(z)^*]^{-1} \\ &= 2[I + G(X)]^{-1}[G(X) + G(z)^*][I + G(z)^*]^{-1} \end{aligned}$$

and, similarly,

$$I - F(X)F(X)^* = 2[I + G(X)]^{-1}[G(X) + G(X)^*][I + G(X)^*]^{-1}$$

for any $X \in [B(\mathcal{H})^n]_1$ and $z \in \mathbb{B}_n$. Using these identities, we deduce that

$$\begin{aligned} [I - F(X)F(z)^*]^{-1}[I - F(X)F(X)^*][I - F(z)F(X)^*]^{-1} \\ = \frac{1}{2}[I + G(z)^*][G(X) + G(z)^*]^{-1}[I + G(X)] \\ \times 2[I + G(X)]^{-1}[G(X) + G(X)^*][I + G(X)^*]^{-1} \\ \times \frac{1}{2}[I + G(X)^*][G(z) + G(X)^*]^{-1}[I + G(z)] \\ = \frac{1}{2}[I + G(z)^*][G(X) + G(z)^*]^{-1}[G(X) + G(X)^*][G(z) + G(X)^*]^{-1}[I + G(z)]. \end{aligned}$$

Now, since

$$\begin{aligned} D_{F(z)^*} &= [I - F(z)F(z)^*]^{1/2} = [2[I + G(z)]^{-1}[G(z) + G(z)^*][I + G(z)^*]^{-1}]^{1/2} \\ &= 2[I + G(z)]^{-1}[\Re G(z)][I + G(z)^*]^{-1}]^{1/2} \end{aligned}$$

the inequality (7.1) implies the inequality of the theorem. The proof is complete. \square

The next result is a Julia type lemma for free holomorphic functions with scalar coefficients and positive real parts.

Theorem 7.3. *Let $G : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$ be a free holomorphic function with $\Re G > 0$. Let $\{z_k\}_{k=1}^\infty \subset \mathbb{B}_n$ be a sequence such that $\lim_{k \rightarrow \infty} z_k = \xi \in \partial \mathbb{B}_n$, $\lim_{k \rightarrow \infty} |G(z_k)| = \infty$, and such that*

$$\lim_{k \rightarrow \infty} \frac{\Re G(z_k)}{(1 - \|z_k\|_2^2)|G(z_k)|^2} = M < \infty.$$

Then $M > 0$ and

$$\Re G(X) \geq \frac{1}{4M} (I - X\xi^*)^{-1} (I - XX^*) (I - \xi X^*)^{-1}$$

for any $X \in [B(\mathcal{H})^n]_1$.

Proof. According to Theorem 7.2, when $\mathcal{E} = \mathbb{C}$, we have

$$\Gamma(z_k) = \frac{2[\Re G(z_k)]^{1/2}}{|1 + G(z_k)|}$$

and

$$\begin{aligned} 4[\Re G(z_k)][G(X) + G(z_k)^*]^{-1}[\Re G(X)][G(z_k) + G(X)^*]^{-1} \\ \geq (1 - \|z_k\|^2)(I - Xz_k^*)^{-1}(I - XX^*)(I - z_k X^*)^{-1}. \end{aligned}$$

Hence, we obtain

$$(7.2) \quad \frac{4\Re G(z_k)}{(1 - \|z_k\|^2)|G(z_k)|^2} \Re G(X) \geq \frac{1}{|G(z_k)|^2} [G(X) + G(z_k)^*] A(k) [G(z_k) + G(X)^*],$$

where $A(k) := (I - Xz_k^*)^{-1}(I - XX^*)(I - z_k X^*)^{-1}$. Taking $X = 0$ in inequality (7.2), we obtain

$$\frac{4\Re G(z_k)}{(1 - \|z_k\|^2)|G(z_k)|^2} \Re G(0) \geq \frac{1}{|G(z_k)|^2} [G(0) + G(z_k)^*] [G(z_k) + G(0)^*],$$

whence

$$\frac{1}{|G(z_k)|^2} [G(0) + G(z_k)^*] [\Re G(0)]^{-1} [G(z_k) + G(0)^*] \leq \frac{\Re G(z_k)}{(1 - \|z_k\|_2^2)|G(z_k)|^2} I.$$

Since $\lim_{k \rightarrow \infty} |G(z_k)| = \infty$ and taking the limit in the latter inequality, we obtain

$$[\Re G(0)]^{-1} \leq \lim_{k \rightarrow \infty} \frac{\Re G(z_k)}{(1 - \|z_k\|_2^2)|G(z_k)|^2} I = MI.$$

Consequently, we have $M > 0$.

Now, due to inequality (7.2) and the fact that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{|G(z_k)|^2} [G(X) + G(z_k)^*] A(k) [G(z_k) + G(X)^*] \\ = \lim_{k \rightarrow \infty} A(k) = (I - X\xi^*)^{-1} (I - XX^*) (I - \xi X^*)^{-1}, \end{aligned}$$

we deduce that

$$4M \Re G(X) \geq (I - X\xi^*)^{-1} (I - XX^*) (I - \xi X^*)^{-1}$$

for any $X \in [B(\mathcal{H})^n]_1$, which completes the proof. \square

We recall (see [37], [39], [44]) that if F is a contractive ($\|F\|_\infty \leq 1$) free holomorphic function with coefficients in $B(\mathcal{E})$, then its boundary function is in $F_n^\infty \otimes B(\mathcal{E})$. Consequently, the evaluation map $\mathbb{B}_n \ni z \mapsto F(z) \in B(\mathcal{E})$ is a contractive operator-valued multiplier of the Drury-Arveson space ([13], [3]), and any such a contractive multiplier has this type of representation.

Corollary 7.4. *The following statements hold.*

- (i) If $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ is a free holomorphic function with $\|F\|_\infty \leq 1$ and $\|F(0)\| < 1$ then

$$\begin{aligned} & [I - F(z)F(w)^*][I - F(w)F(w)^*]^{-1}[I - F(w)F(z)^*] \\ & \leq \frac{|(1 - \langle w, z \rangle)|^2}{(1 - \|z\|^2)(1 - \|w\|^2)}[I - F(z)F(z)^*]. \end{aligned}$$

for any $z, w \in \mathbb{B}_n$.

- (ii) If $G : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$ is a free holomorphic function with $\Re G > 0$, then

$$[G(z) + G(w)^*][\Re G(w)]^{-1}[G(w) + G(z)^*] \leq \frac{4|(1 - \langle w, z \rangle)|^2}{(1 - \|z\|^2)(1 - \|w\|^2)}\Re G(z)$$

for any $z, w \in \mathbb{B}_n$.

Proof. Taking $X = w \in \mathbb{B}_n$ in Theorem 7.1, we deduce that

$$\begin{aligned} & [I - F(w)F(z)^*]^{-1}[I - F(w)F(w)^*][I - F(z)F(w)^*]^{-1} \\ & \geq \frac{(1 - \|z\|_2^2)(1 - \|w\|_2^2)}{|1 - \langle w, z \rangle|^2}[I - F(w)F(w)^*]^{-1} \end{aligned}$$

for any $z, w \in \mathbb{B}_n$. We recall that if A, B are two positive invertible operator such that $A \leq B$ then $B^{-1} \leq A^{-1}$. Applying this result to the inequality above, we complete the proof of item (i).

To prove part (ii), take $X = w \in \mathbb{B}_n$ in Theorem 7.2. We obtain

$$\begin{aligned} & [I + G(z)^*][G(w) + G(z)^*]^{-1}[\Re G(w)][G(z) + G(w)^*]^{-1}[I + G(z)] \\ & \geq \frac{(1 - \|z\|_2^2)(1 - \|w\|^2)}{4|1 - \langle w, z \rangle|^2}[I + G(z)^*][\Re G(z)]^{-1}[I + G(z)] \end{aligned}$$

Multiplying to the left by $[I + G(z)^*]^{-1}$ and to the right by $[I + G(z)]^{-1}$, and passing to inverses, as above, we obtain the desired inequality. The proof is complete. \square

8. LINDELÖF INEQUALITY AND SHARPENED FORMS OF THE NONCOMMUTATIVE VON NEUMANN INEQUALITY

In this section, we provide a noncommutative generalization of a classical inequality due to Lindelöf, which turns out to be sharper than the noncommutative von Neumann inequality.

Theorem 8.1. *If $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ is a free holomorphic function, then*

$$\|F(X)\| \leq \frac{\|X\| + \|F(0)\|}{1 + \|X\|\|F(0)\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

If, in addition, the boundary function of F has its entries in the noncommutative disc algebra \mathcal{A}_n , then the inequality above can be extended to any $X \in [B(\mathcal{H})^n]_1^-$.

Proof. First, we consider the case when $\|F(0)\| < 1$. Using the first inequality of Corollary 6.2, in the particular case when $a = 0$, we obtain

$$\Delta_{F(0)}[I - F(X)F(0)^*]^{-1}[I - F(X)F(X)^*][I - F(0)F(X)^*]^{-1}\Delta_{F(0)} \geq I - XX^*.$$

Hence, we deduce that

$$(8.1) \quad I - F(X)F(X)^* \geq \frac{1 - \|X\|^2}{1 - \|F(0)\|^2}[I - F(X)F(0)^*][I - F(0)F(X)^*].$$

On the other hand, since $\|F(0)\| < 1$, the operator $I - F(X)F(0)^*$ is invertible and

$$\begin{aligned} \|[I - F(X)F(0)^*]^{-1}\| & \leq 1 + \|F(X)\|\|F(0)\| + \|F(X)\|^2\|F(0)\|^2 + \dots \\ & = \frac{1}{1 - \|F(X)\|\|F(0)\|}. \end{aligned}$$

Similarly, we have $\|[I - F(0)F(X)^*]^{-1}\| \leq \frac{1}{1 - \|F(X)\|\|F(0)\|}$ and deduce that

$$\begin{aligned} [I - F(0)F(X)^*]^{-1}[I - F(X)F(0)^*]^{-1} &\leq \|[I - F(0)F(X)^*]^{-1}\| \|[I - F(X)F(0)^*]^{-1}\| I \\ &\leq \frac{1}{(1 - \|F(X)\|\|F(0)\|)^2} I. \end{aligned}$$

Hence, we obtain

$$[I - F(X)F(0)^*][I - F(0)F(X)^*] \geq (1 - \|F(X)\|\|F(0)\|)^2 I,$$

which combined with inequality (8.1), leads to

$$F(X)F(X)^* \leq \left(1 - \frac{1 - \|X\|^2}{1 - \|F(0)\|^2} (1 - \|F(X)\|\|F(0)\|)^2\right) I.$$

This inequality implies

$$\|F(X)\|^2 \leq 1 - \frac{1 - \|X\|^2}{1 - \|F(0)\|^2} (1 - \|F(X)\|\|F(0)\|)^2,$$

which is equivalent to

$$(1 - \|F(X)\|^2)(1 - \|F(0)\|^2) \geq (1 - \|X\|^2)(1 - \|F(X)\|\|F(0)\|)^2.$$

Straightforward calculations show that the latter inequality is equivalent to

$$(\|F(X)\| - \|F(0)\|)^2 \leq \|X\|^2 (1 - \|F(X)\|\|F(0)\|)^2.$$

Hence, we obtain

$$\|F(X)\| - \|F(0)\| \leq \|X\| - \|X\|\|F(X)\|\|F(0)\|,$$

which is equivalent to

$$\|F(X)\| \leq \frac{\|X\| + \|F(0)\|}{1 + \|X\|\|F(0)\|}, \quad X \in [B(\mathcal{H})^n]_1.$$

Now, we consider the case when $\|F(0)\| = 1$. Applying our result above to ϵF , where $\epsilon \in (0, 1)$, we get

$$\epsilon \|F(X)\| \leq \frac{\|X\| + \epsilon \|F(0)\|}{1 + \epsilon \|X\|\|F(0)\|}.$$

Taking $\epsilon \rightarrow 0$, the result follows. Now, consider the case when the boundary function of F has its entries in the noncommutative disc algebra \mathcal{A}_n . According to [39], we have

$$F(X) = \lim_{r \rightarrow 1} F(rX_1, \dots, rX_n), \quad X = (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1^-,$$

in the operator norm topology. Applying inequality of this theorem to the free holomorphic function $X \mapsto F(rX_1, \dots, rX_n)$ and taking $r \rightarrow 1$, we complete the proof. \square

A few remarks are necessary. First, notice that in the particular case when $F(0) = 0$, Theorem 8.1 implies the noncommutative Schwarz type result. We also remark that if $\|F(0)\| < 1$, then

$$\frac{\|X\| + \|F(0)\|}{1 + \|X\|\|F(0)\|} < 1, \quad X \in [B(\mathcal{H})^n]_1.$$

Therefore, the inequality in Theorem 8.1 is sharper than the noncommutative von Neumann inequality, which gives only $\|F(X)\| \leq 1$, when $\|F\|_\infty \leq 1$.

We recall that if $F := (F_1, \dots, F_m)$ is a contractive ($\|F\|_\infty \leq 1$) free holomorphic function, then the evaluation map $\mathbb{B}_n \ni z \mapsto F(z) \in \mathbb{B}_m$ is a contractive matrix-valued multiplier of the Drury-Arveson space and, moreover, any such a contractive multiplier has this kind of representation. In particular, Theorem 8.1 implies that

$$\|F(z)\| \leq \frac{\|z\| + \|F(0)\|}{1 + \|z\|\|F(0)\|}, \quad z \in \mathbb{B}_n.$$

for any contractive multiplier $F : \mathbb{B}_n \rightarrow \mathbb{B}_m$ of the Drury-Arveson space.

We consider now the particular case when $m = 1$. Here is a sharpened form of the noncommutative von Neumann inequality (see [34])

Corollary 8.2. *If $f : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H})$ is a nonconstant free holomorphic function with $\|f\|_\infty \leq 1$, then*

$$\|f(X)\| \leq \frac{\|X\| + |f(0)|}{1 + \|X\| |f(0)|} < 1, \quad X \in [B(\mathcal{H})^n]_1.$$

If, in addition, f is in the noncommutative disc algebra \mathcal{A}_n , then the left inequality holds for any $X \in [B(\mathcal{H})^n]_1^-$.

Another consequence of Theorem 8.1 is the following.

Corollary 8.3. *Let $F : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$ be a free holomorphic function and let $z \in \mathbb{B}_n$, then*

$$\|F(X)\| \leq \frac{\|\Phi_z(X)\| + \|F(z)\|}{1 + \|\Phi_z(X)\| \|F(z)\|}, \quad X \in [B(\mathcal{H})^n]_1,$$

where Φ_z is the free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^n]_1$ associated $z \in \mathbb{B}_n$.

Proof. Applying Theorem 8.1 to the free holomorphic function $F \circ \Phi_z : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^m]_1^-$, we obtain

$$\|(F \circ \Phi_z)(Y)\| \leq \frac{\|Y\| + \|(F \circ \Phi_z)(0)\|}{1 + \|Y\| \|(F \circ \Phi_z)(0)\|}, \quad Y \in [B(\mathcal{H})^n]_1.$$

Taking into account that $\Phi_z(0) = z$, $\Phi_z \circ \Phi_z = \text{id}$, and setting $Y = \Phi_z(X)$, $X \in [B(\mathcal{H})^n]_1$, in the inequality above, we obtain the desired inequality. \square

9. PSEUDOHYPERBOLIC METRIC ON THE UNIT BALL OF $B(\mathcal{H})^n$ AND AN INVARIANT SCHWARZ-PICK LEMMA

The pseudohyperbolic distance on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane is defined by

$$d_1(z, w) := \left| \frac{z - w}{1 - \bar{z}w} \right|, \quad z, w \in \mathbb{D}.$$

Some of the basic properties of the pseudohyperbolic distance are the following:

- (i) the pseudohyperbolic distance is invariant under the conformal automorphisms of \mathbb{D} , i.e.,

$$d_1(\varphi(z), \varphi(w)) = d_1(z, w), \quad z, w \in \mathbb{D},$$

for all $\varphi \in \text{Aut}(\mathbb{D})$;

- (ii) the d_1 -topology induced on the open disc is the usual planar topology;
- (iii) any analytic function $f : \mathbb{D} \rightarrow \mathbb{D}$ is distance-decreasing, i.e., satisfies

$$d_1(f(z), f(w)) \leq d_1(z, w), \quad z, w \in \mathbb{D}.$$

The analogue of the pseudohyperbolic distance for the open unit ball of \mathbb{C}^n ,

$$\mathbb{B}_n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|_2 < 1\},$$

is defined by

$$d_n(z, w) = \|\psi_z(w)\|_2, \quad z, w \in \mathbb{B}_n,$$

where ψ_z is the involutive automorphism of \mathbb{B}_n that interchanges 0 and z . This distance has properties similar to those of d_1 (see [51], [58]).

In what follows, we introduce a pseudohyperbolic metric on the noncommutative ball $[B(\mathcal{H})^n]_1$, which satisfies properties similar to those of the pseudohyperbolic metric d_1 on the unit disc \mathbb{D} and which is a noncommutative extension of d_n on the open unit ball of \mathbb{C}^n . In particular, we obtain a Schwarz-Pick lemma for free holomorphic functions on $[B(\mathcal{H})^n]_1$ with respect to this pseudohyperbolic metric.

We recall ([47]) that $A, B \in [B(\mathcal{H})^n]_1^-$ are called Harnack equivalent (and denote $A \stackrel{H}{\sim}_c B$) if and only if there exists $c \geq 1$ such that

$$\frac{1}{c^2} \operatorname{Re} p(B_1, \dots, B_n) \leq \operatorname{Re} p(A_1, \dots, A_n) \leq c^2 \operatorname{Re} p(B_1, \dots, B_n)$$

for any noncommutative polynomial with matrix-valued coefficients $p \in \mathbb{C}[X_1, \dots, X_n] \otimes M_{m \times m}$, $m \in \mathbb{N}$, such that $\operatorname{Re} p \geq 0$. The equivalence classes with respect to $\stackrel{H}{\sim}$ are called Harnack parts of $[B(\mathcal{H})^n]_1^-$. We proved that the open unit ball $[B(\mathcal{H})^n]_1$ is a distinguished Harnack part of $[B(\mathcal{H})^n]_1^-$, namely, the Harnack part of 0.

Now, let Δ be a Harnack part of $[B(\mathcal{H})^n]_1^-$ and define the map $\mathbf{d} : \Delta \times \Delta \rightarrow [0, \infty)$ by setting

$$(9.1) \quad \mathbf{d}(A, B) := \frac{\omega(A, B)^2 - 1}{\omega(A, B)^2 + 1}, \quad A, B \in \Delta.$$

where

$$\omega(A, B) := \inf \left\{ c \geq 1 : A \stackrel{H}{\sim}_c B \right\}, \quad A, B \in \Delta.$$

The first result of this section is the following.

Theorem 9.1. *Let Δ be a Harnack part of $[B(\mathcal{H})^n]_1^-$. Then the map \mathbf{d} defined by relation (9.1) has the following properties:*

- (i) \mathbf{d} is a bounded metric on Δ ;
- (ii) for any free holomorphic automorphism Φ of the noncommutative unit ball $[B(\mathcal{H})^n]_1$,

$$\mathbf{d}(X, Y) = \mathbf{d}(\Phi(X), \Phi(Y)), \quad X, Y \in \Delta.$$

Proof. According to Lemma 2.1 from [47], if Δ is a Harnack part of $[B(\mathcal{H})^n]_1^-$ and $A, B, C \in \Delta$, then the following properties hold:

- (a) $\omega(A, B) \geq 1$ and $\omega(A, B) = 1$ if and only if $A = B$;
- (b) $\omega(A, B) = \omega(B, A)$;
- (c) $\omega(A, C) \leq \omega(A, B)\omega(B, C)$.

Part (c) can be used to show that

$$\mathbf{d}(A, C) \leq \mathbf{d}(A, B) + \mathbf{d}(B, C).$$

Indeed, define the function $f : [1, \infty) \rightarrow [0, \infty)$ by $f(x) := \frac{x^2 - 1}{x^2 + 1}$. Since $f'(x) = \frac{2x}{(x^2 + 1)^2} \geq 0$, we deduce that f is increasing. Hence, and due to inequality (c), we have

$$f(\omega(A, C)) \leq f(\omega(A, B)\omega(B, C)).$$

Since $f(\omega(A, C)) = \mathbf{d}(A, C)$, it remains to prove that

$$f(\omega(A, B)\omega(B, C)) \leq f(\omega(A, B)) + f(\omega(B, C)).$$

Setting $x := \omega(A, B)$ and $y := \omega(B, C)$, the inequality above is equivalent to

$$\frac{x^2 y^2 - 1}{x^2 y^2 + 1} \leq \frac{x^2 - 1}{x^2 + 1} + \frac{y^2 - 1}{y^2 + 1}.$$

Straightforward calculations reveal that the latter inequality is equivalent to

$$(x^2 y^2 - 1)(x^2 - 1)(y^2 - 1) \geq 0,$$

which holds for any $x, y \geq 1$. Using (a) and (b), one can deduce that \mathbf{d} is a metric.

Now, we prove part (ii). According to Lemma 2.3 from [47], if A and B are in $[B(\mathcal{H})^n]_1^-$, $c \geq 1$, and $\Psi \in \operatorname{Aut}([B(\mathcal{H})^n]_1)$, then $A \stackrel{H}{\sim}_c B$ if and only if $\Phi(A) \stackrel{H}{\sim}_c \Phi(B)$. Consequently, if $A, B \in \Delta$, then we deduce that $\omega(A, B) = \omega(\Phi(A), \Phi(B))$, which implies (ii). The proof is complete. \square

We introduced in [47] a hyperbolic (*Poincaré-Bergman* [7] type) metric δ on any Harnack part Δ of $[B(\mathcal{H})^n]_1^-$ by setting

$$(9.2) \quad \delta(A, B) := \ln \omega(A, B).$$

We will use the properties of δ to deduce the following result concerning the pseudohyperbolic distance on the open noncommutative ball $[B(\mathcal{H})^n]_1$.

Theorem 9.2. *The pseudohyperbolic metric $\mathbf{d} : [B(\mathcal{H})^n]_1 \times [B(\mathcal{H})^n]_1 \rightarrow [0, \infty)$ has the following properties:*

(i) *for any $X, Y \in [B(\mathcal{H})^n]_1$,*

$$\mathbf{d}(X, Y) = \tanh \delta(X, Y).$$

(ii) *$\mathbf{d}|_{\mathbb{B}_n \times \mathbb{B}_n}$ coincides with the pseudohyperbolic distance on \mathbb{B}_n , i.e.,*

$$\mathbf{d}(z, w) = \|\psi_z(w)\|_2, \quad z, w \in \mathbb{B}_n,$$

where ψ_z is the involutive automorphism of \mathbb{B}_n that interchanges 0 and z .

(iii) *the \mathbf{d} -topology coincides with the norm topology on the open unit ball $[B(\mathcal{H})^n]_1$;*

(iv) *for any $X, Y \in [B(\mathcal{H})^n]_1$,*

$$\mathbf{d}(X, Y) := \frac{\max\{\|\Gamma\|, \|\Gamma^{-1}\|\} - 1}{\max\{\|\Gamma\|, \|\Gamma^{-1}\|\} + 1},$$

where

$$\Gamma := (C_X C_Y^{-1})^* (C_X C_Y^{-1}), \quad C_X := (\Delta_X \otimes I)(I - R_X)^{-1},$$

and $R_X := X_1^ \otimes R_1 + \cdots + X_n^* \otimes R_n$ is the reconstruction operator.*

Proof. Part (i) follows from relations (9.1) and (9.2). Part (ii) follows from part (i) and the fact that, according to [47], $\delta|_{\mathbb{B}_n \times \mathbb{B}_n}$ coincides with the Poincaré-Bergman distance on \mathbb{B}_n , i.e.,

$$\delta(z, w) = \frac{1}{2} \ln \frac{1 + \|\psi_z(w)\|_2}{1 - \|\psi_z(w)\|_2}, \quad z, w \in \mathbb{B}_n,$$

where ψ_z is the involutive automorphism of \mathbb{B}_n that interchanges 0 and z .

Since the δ -topology coincides with the norm topology on the open unit ball $[B(\mathcal{H})^n]_1$, part (i) implies (iii). In [47], we proved that

$$\delta(A, B) = \ln \max \{ \|C_A C_B^{-1}\|, \|C_B C_A^{-1}\| \}, \quad A, B \in [B(\mathcal{H})^n]_1,$$

where $C_X := (\Delta_X \otimes I)(I - R_X)^{-1}$ and $R_X := X_1^* \otimes R_1 + \cdots + X_n^* \otimes R_n$ is the reconstruction operator. Hence and due to (i), part (iv) follows. \square

We showed in [47] that if $A, B \in [B(\mathcal{H})^n]_1^-$, then $A \stackrel{H}{\sim} B$ if and only if $rA \stackrel{H}{\sim} rB$ for any $r \in [0, 1)$ and $\sup_{r \in [0, 1)} \omega(rA, rB) < \infty$. Moreover, in this case, the function $r \mapsto \omega(rA, rB)$ is ancreasing on $[0, 1)$ and $\omega(A, B) = \sup_{r \in [0, 1)} \omega(rA, rB)$. As a consequence, one can see that if $A \stackrel{H}{\sim} B$, then the function $r \mapsto \mathbf{d}(rA, rB)$ are increasing on $[0, 1)$ and $\mathbf{d}(A, B) = \sup_{r \in [0, 1)} \mathbf{d}(rA, rB)$.

This result together with Theorem 9.2 can be used to obtain an explicit formula for the pseudohyperbolic metric on any Harnack part of $[B(\mathcal{H})^n]_1^-$.

Now we provide a Schwarz-Pick lemma for free holomorphic functions on $[B(\mathcal{H})^n]_1$ with operator-valued coefficients, with respect to the pseudohyperbolic metric.

Theorem 9.3. *Let $F_j : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E})$, $j = 1, \dots, m$, be free holomorphic functions with coefficients in $B(\mathcal{E})$, and assume that $F := (F_1, \dots, F_m)$ is a contractive free holomorphic function. If $X, Y \in [B(\mathcal{H})^n]_1$, then $F(X) \stackrel{H}{\sim} F(Y)$ and*

$$\mathbf{d}(F(X), F(Y)) \leq \mathbf{d}(X, Y),$$

where \mathbf{d} is the pseudohyperbolic metric defined on the Harnack parts of $[B(\mathcal{H})^n]_1^-$.

Proof. According to the proof of Theorem 4.2 from [47], we have $F(X) \stackrel{H}{\sim} F(Y)$ and

$$\omega(F(X), F(Y)) \leq \omega(X, Y).$$

Using the definition (9.1) and the fact that the function $f(x) := \frac{x^2-1}{x^2+1}$ is increasing on the interval $[1, \infty)$, the result follows. \square

If $F := (F_1, \dots, F_m)$ is a contractive ($\|F\|_\infty \leq 1$) free holomorphic function with coefficients in $B(\mathcal{E})$, then the evaluation map $\mathbb{B}_n \ni z \mapsto F(z) \in B(\mathcal{E})^{(m)}$ is a contractive operator-valued multiplier of the Drury-Arveson space, and any such a contractive multiplier has this type of representation.

Corollary 9.4. *Let $F := (F_1, \dots, F_m)$ be a contractive free holomorphic function with coefficients in $B(\mathcal{E})$. If $z, w \in \mathbb{B}_n$, then $F(z) \stackrel{H}{\sim} F(w)$ and*

$$\mathbf{d}(F(z), F(w)) \leq \mathbf{d}(z, w).$$

REFERENCES

- [1] D. ALPAY AND D.S. KALYUZHNYI-VERBOVETSKII, Matrix- J -unitary non-commutative rational formal power series, in: *The state space method generalizations and applications*, pp. 49–113, Oper. Theory Adv. Appl., **161**, Birkhuser, Basel, 2006.
- [2] T. ANDO AND K. FAN, Pick-Julia theorems for operators, *Math. Z.* **168** (1979), 23–34.
- [3] W.B. ARVESON, Subalgebras of C^* -algebras III: Multivariable operator theory, *Acta Math.* **181** (1998), 159–228.
- [4] J.A. BALL AND V. BOLOTNIKOV, On bitangential interpolation problem for contractive-valued functions on the unit ball, *Linear Algebra Appl.* **353** (2002), 107–147.
- [5] J. A. BALL, G. GROENEWALD, AND T. MALAKORN, Conservative structured noncommutative multidimensional linear systems, *The state space method generalizations and applications*, 179–223, Oper. Theory Adv. Appl., **161**, Birkhuser, Basel, 2006.
- [6] J. A. BALL, G. GROENEWALD, AND T. MALAKORN, Bounded real lemma for structured noncommutative multidimensional linear systems and robust control, *Multilinear Systems and Signal Processing* **17** (2006), 119–150.
- [7] S. BERGMAN, *The kernel function and conformal mapping*, Mathematical Surveys, No. V. American Mathematical Society, Providence, R.I., 1970. x+257 pp.
- [8] C. CARATHÉODORY, Über die Winkelderivierten von beschränkten analytischen Funktionen, *Sitzungsberg Preuss. Acad.Phys.-Math.* **4**, 1–18.
- [9] C. CARATHÉODORY, *Theory of Functions of a Complex Variable*, Chelsea Publishing Co., 1954.
- [10] J.B. CONWAY, *Functions of one complex variable. I*. Second Edition. Graduate Texts in Mathematics **159**. Springer-Verlag, New York, 1995.
- [11] J. CUNTZ, Simple C^* -algebras generated by isometries, *Commun. Math. Phys.* **57** (1977), 173–185.
- [12] K. R. DAVIDSON Free semigroup algebras. A survey. *Systems, approximation, singular integral operators, and related topics* (Bordeaux, 2000), 209–240, Oper. Theory Adv. Appl., **129**, Birkhuser, Basel, 2001.
- [13] S. DRURY, A generalization of von Neumann’s inequality to the complex ball, *Proc. Amer. Math. Soc.* **68** (1978), 300–304.
- [14] K. FAN, Analytic functions of a proper contraction, *Math. Z* **160** (1978), 275–290.
- [15] K. FAN, Julia’s lemma for operators, *Math. Ann.* **239** (1979), 241–245.
- [16] J. GARNETT, *Bounded analytic functions*, Academic Press, New York, 1981.
- [17] J. GLICKSBERG, Julia’s lemma for function algebras, *Duke Math. J.* **43**(1976), 277–284.
- [18] L.A. HARRIS, Schwarz’s lemma in normed linear spaces, *Proc. Nat. Acad. Sci. USA* **62** (1969), 1014–1017.
- [19] L.A. HARRIS, Bounded symmetric homogeneous domains in infinite dimensional spaces, *Proceedings on Infinite Dimensional Holomorphy (Internat. Conf., Univ. Kentucky, Lexington, Ky., 1973)*, pp. 13–40. *Lecture Notes in Math.*, Vol. 364, Springer, Berlin, 1974.
- [20] J.W. HELTON, I. KLEP, S. MCCULLOUGH, AND N. SLINGLED, Noncommutative ball maps, *J. Funct. Anal.* **257** (2009), 47–87.
- [21] E. HILLE, *Analytic function theory II*, Boston: Ginn 1962.
- [22] D. S. KALIUZHNYI-VERBOVETSKYI, Carathéodory interpolation on the non-commutative polydisk, *J. Funct. Anal.* **229** (2005), no. 2, 241–276.
- [23] D. S. KALIUZHNYI-VERBOVETSKYI AND V. VINNIKOV, Singularities of rational functions and minimal factorizations: the noncommutative and the commutative setting, *Linear Algebra Appl.* **430** (2009), no. 4, 869–889.
- [24] S.G. KRANTZ, *Geometric function theory*, Explorations in complex analysis. Cornerstones. Birkhuser Boston, Inc., Boston, MA, 2006. xiv+314 pp.
- [25] G. JULIA, Extension d’un lemme de Schwarz, *Acta Math.* **42** (1920), 349–355.
- [26] P.S. MUHLY AND B. SOLEL, Tensor algebras over C^* -correspondences: representations, dilations, and C^* -envelopes, *J. Funct. Anal.* **158** (1998), 389–457.

- [27] P.S. MUHLY AND B. SOLEL, Hardy algebras, W^* -correspondences and interpolation theory, *Math. Ann.* **330** (2004), 353–415.
- [28] P.S. MUHLY AND B. SOLEL, Schur Class Operator Functions and Automorphisms of Hardy Algebras, *Documenta Math.* **13** (2008), 365–411.
- [29] V.I. PAULSEN, G. POPESCU, AND D. SINGH, On Bohr’s inequality, *Proc. London Math. Soc.* **85** (2002), 493–512.
- [30] R.S. PHILLIPS, On symplectic mappings of contraction operators, *Studia Math.* **31**(1968), 15–27.
- [31] G. PICK, Über die Beschränkungen analytische Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden, *Math. Ann.* (1916) **77**, 7–23.
- [32] G. POPESCU, Isometric dilations for infinite sequences of noncommuting operators, *Trans. Amer. Math. Soc.* **316** (1989), 523–536.
- [33] G. POPESCU, Characteristic functions for infinite sequences of noncommuting operators, *J. Operator Theory* **22** (1989), 51–71.
- [34] G. POPESCU, Von Neumann inequality for $(B(H)^n)_1$, *Math. Scand.* **68** (1991), 292–304.
- [35] G. POPESCU, Functional calculus for noncommuting operators, *Michigan Math. J.* **42** (1995), 345–356.
- [36] G. POPESCU, Multi-analytic operators on Fock spaces, *Math. Ann.* **303** (1995), 31–46.
- [37] G. POPESCU, Poisson transforms on some C^* -algebras generated by isometries, *J. Funct. Anal.* **161** (1999), 27–61.
- [38] G. POPESCU, Curvature invariant for Hilbert modules over free semigroup algebras, *Adv. Math.* **158** (2001), 264–309.
- [39] G. POPESCU, Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$, *J. Funct. Anal.* **241** (2006), 268–333.
- [40] G. POPESCU, Multivariable Bohr inequalities, *Trans. Amer. Math. Soc.*, **359** (2007), 5283–5317.
- [41] G. POPESCU, Free holomorphic functions and interpolation, *Math. Ann.*, **342** (2008), 1–30.
- [42] G. POPESCU, Free pluriharmonic majorants and commutant lifting, *J. Funct. Anal.* **255** (2008), 891–939.
- [43] G. POPESCU, Hyperbolic geometry on the unit ball of $B(\mathcal{H})^n$ and dilation theory, *Indiana Univ. Math. J.* **57** (2008), No.6, 2891–2930.
- [44] G. POPESCU, Noncommutative transforms and free pluriharmonic functions, *Adv. Math.* **220** (2009), 831–893.
- [45] G. POPESCU, Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$, *J. Reine Angew. Math.*, to appear.
- [46] G. POPESCU, Unitary invariants in multivariable operator theory, *Mem. Amer. Math. Soc.*, **200** (2009), No.941, 91 pages.
- [47] G. POPESCU, Noncommutative hyperbolic geometry on the unit ball of $B(H)^n$, *J. Funct. Anal.* **256** (2009), 4030–4070.
- [48] G. POPESCU, Operator theory on noncommutative domains, *Mem. Amer. Math. Soc.*, to appear, 124 pages.
- [49] V.P. POTAPOV, The multiplicative structure of J -contractive matrix functions, *Amer. Math. Soc. Transl.*(2) **15** (1960), 131–243.
- [50] W. RUDIN, *Real and Complex Analysis*, McGraw-Hill Book Co. (1966)
- [51] W. RUDIN, *Function theory in the unit ball of \mathbb{C}^n* , Springer-verlag, New-York/Berlin, 1980.
- [52] C.L. SIEGEL, Symplectic geometry, *Amer. J. Math.* **65** (1943), 1–86.
- [53] E.C. TITCHMARSH, *The Theory of Functions*, Oxford University Press, Oxford, 1939.
- [54] B. SZ.-NAGY AND C. FOIAŞ, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York 1970.
- [55] D.V. VOICULESCU Free analysis questions II: The Grassmannian completion and the series expansions at the origin, preprint.
- [56] J. VON NEUMANN, Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.* **4** (1951), 258–281.
- [57] N.J. YOUNG, Orbits of the unit sphere of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ under symplectic transformations, *J. Operator Theory* **11** (1984), 171–191.
- [58] K. ZHU, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005. x+271 pp.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT SAN ANTONIO, SAN ANTONIO, TX 78249, USA

E-mail address: gelu.popescu@utsa.edu